SOME ASPECTS OF THE PROBLEM OF MATHEMATICAL RIGOR[1]

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1. **Introduction.** The cue for the title to this address is taken from that of one by Pierpont before the Nashville meeting of this Society several years ago.[2] This is typical of a number of expository treatments of this topic which have been presented to the mathematical public in recent years.[3] In the present paper I shall discuss the same theme in a somewhat different manner. Relying upon these expository addresses for the historical background, I propose to treat certain aspects of the subject which have been rather neglected in them. The discussion is frankly from a single point of view, which is a species of formalism. I shall try, in the first place, to explain the fundamental concepts of formalism, and, in the second place, to add some new suggestions and criticisms in matters of detail.[4]

The problem of mathematical rigor is that of giving an objective definition of a rigorous proof. If you will examine your ideas on this subject I think you will agree that there is something vague and subjective about them. This does not mean, of course, that they are unsatisfactory for the needs of working mathematicians. In daily life, when we say that a piece of cloth is a yard wide, we really mean that its width is a certain legally defined fraction of the distance between two scratches on a metal bar located in Paris; nevertheless we do not rush to Paris when we wish to verify that a piece of cloth has this property. Secondary standards of varying degrees of accuracy suffice for the needs of daily life and of science; but neither science nor business would be possible without exact primary standards. Even so we need a primary standard of rigor in mathematics. The definition of such a standard, and the elaboration of practical secondary standards

[1] An address delivered before the meeting of the Society in New York City on October 26, 1940, by invitation of the Program Committee.

[2] J. Pierpont, *Mathematical rigor, past and present*, this Bulletin, vol. 34 (1928), pp. 23–53.

[3] See for example A. Dresden, *Some philosophical aspects of mathematics*, this Bulletin, vol. 34 (1928), pp. 438–452; G. H. Hardy, *Mathematical proof*, Mind, vol. 38 (1929), pp. 1–25; E. R. Hedrick, *Tendencies in the logic of mathematics*, Science, vol. 77 (1933), pp. 335–343.

[4] For views related to those here presented, see my paper *Remarks on the definition and nature of mathematics*, Journal of Unified Science, vol. 9, pp. 164–169. This is an abstract of an address delivered before the Fifth International Congress of Unified Science at Cambridge, Massachusetts, September 5, 1939.

in relation to it, constitutes the outstanding problem of the foundations of mathematics.

This problem is evidently tantamount to finding exact criteria for the truth of a mathematical proposition. For we should regard a rigorous proof as a process of making manifest the truth of its conclusion; and the problem of rigor is inseparable from that of truth.

2. **Critique of the non-formal theories of mathematics.** There are three main types of view as to the nature of mathematical truth; these I shall call realism, idealism, and formalism. The first of these is the view that mathematical propositions concern the real world (in the sense of our physical environment); the second holds that mathematics deals with mental objects of some sort. These two views have in common the notion that mathematical propositions have an essential subject-matter or content, with respect to which truth and rigor are definable; they are what the Germans would call "inhaltlich." Since there is no satisfactory translation of this word into English,[5] I propose that we call them contensive, where 'contensive' is derived from 'content' in the same way as 'intensive' from 'intent.' The formalist view will form the main subject of the present paper; but before we take it up it will be appropriate to consider a few remarks about the others.

The realist view evidently does not need to be taken seriously at the present time. Of course it is the original view—the mathematics of primitive peoples is essentially empirical—and it is tenable for simple arithmetic propositions such as a $2+2=4$. But that the infinitistic conceptions of modern mathematics have no counterpart in the external environment is a point which needs no elaboration.

The idealist view has many varieties according to the nature of the mental objects on which it is based. On the one hand there is the view, called by Bernays[6] Platonism, which ascribes an independent existence to all the infinitistic conceptions of mathematics; at the other extreme there is intuitionism, which denies this existence, but bases everything on an a priori intuition. Whatever the nature of these mental objects, all forms of idealism are subject to the same fundamental criticism, namely, that the criterion of truth is vague at best, and the existence of the mental objects is a metaphysical hypoth-

[5] Translations sometimes used are 'material' and 'intuitive'; but these have undesirable connotations.

[6] P. Bernays, *Sur le platonisme dans les mathématiques*, L'Enseignement Mathématique, vol. 34 (1935–1936), pp. 52–69. There are naturally many different varieties of Platonism and also, for that matter, of intuitionism.

esis from which mathematics should be free, if it is to be an objective science.

That this criticism applies to Platonism has been shown by its intuitionist critics, and is now fairly generally admitted. But it also applies to more moderate forms of idealism, and even to intuitionism itself. For the vagueness of the fundamental "oerintuitie der wiskunde" has been noticed by several persons; and Heyting himself, in his Ergebnisse report of 1934, explicitly denies the possibility of describing it exactly.[7] On the other hand it is clear from the intuitionists' writings that they conceive their fundamental intuition to be (1) essentially a thinking activity,[8] (2) a priori,[9] (3) independent of language,[10] and (4) objective, in that it is the same in all thinking beings.[11] The existence of an intuition having these properties—or even the first three of them—is an outright assumption. It is doubtful, indeed, if there is any a priori knowledge; and it has been rationally maintained that thinking of any kind is impossible without language. Furthermore the fourth property, although it is absolutely vital if intuitionism is to give any account of mathematical truth at all, seems patently incompatible with the other three. In short, the intuitionist definition of mathematics has meaning only for one who postulates an a priori intuition which is both objective and pre-linguistic. Although such a postulate is agreeable to certain types of philosophy, yet it is an assumption for all that; and one which, from other points of view, is highly dubious and metaphysical.

The intuitionist school has made contributions of great value to the foundations of mathematics. Not only have they cleared away much of the rubbish of old-fashioned Platonism, but they have helped to sharpen the ideas of the formalists. We shall see later that most of the constructive suggestions of the intuitionists, when cleared of their metaphysics, find their justification in modern formalism. But

[7] A. Heyting, *Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie*, Ergebnisse der Mathematik und ihre Grenzgebiete, vol. 3, no. 4, Berlin, Springer, 1934. See for example, page 12 where he writes "Überdies ist es an sich widersinnig, die Möglichkeiten des Denkens in das Mieder bestimmter zuvor angebener Konstruktionsprinzipien zwängen zu wollen. Man muss sich also darauf beschränken, durch mehr oder weniger vage Umschreibungen in dem Hörer die mathematische Geisteshaltung hervorzurufen."

[8] "Eine konstruktive Tätigkeit unseres Verstandes" (Heyting, loc. cit., p. 2).

[9] "Die mathematischen Gegenstände werden von dem denkenden Geist unmittelbar erfasst; die mathematische Erkenntnis ist daher von der Erfahrung unabhängig." (Heyting, loc. cit., p. 3.)

[10] Cf. Heyting, loc. cit., p. 13.

[11] A fifth characteristic, its relation to time, is irrelevant for us.

the fundamental basis of their theory is a metaphysical doctrine.

Before leaving the subject of contensive definitions it will be necessary to consider the term *logicism* (or logisticism). It is sometimes maintained that there are three principal views as to the foundations of mathematics: intuitionism, formalism, and logicism, where the last is the view that mathematics is reducible to logic. If we had a satisfactory definition of logical truth and of logical rigor, logicism would solve our problem very nicely. But the paradoxes have shown that this is not the case; and in fact logic is as much in need of definition as mathematics. Hence to say that mathematics is logic is merely to replace one undefined term by another. Practically all competent writers realize this; and those who maintain that mathematics is logic generally couple this contention with some explanation of what logic is. But when we take these definitions of logic into account, the resulting definitions of mathematics are not a unified view according to the classification here adopted. Ramsey was a Platonist; Frege was essentially a formalist, and most of the modern writers who are popularly classified as logicists are definitely so. 'Logicism,' then, is not the name of a distinct view in regard to the foundations of mathematics.

From these considerations we see that a satisfactory contensive definition of mathematical truth and rigor has not been made; and indeed there is reasonable doubt as to whether such a definition is possible. Although such conceptions are current among mathematicians, they are useful only as secondary standards.

3. **The notion of formal system.** This brings us to formalism. According to this view the emphasis is not on the content of mathematical propositions, but on the method of derivation. We have an exact definition of a rigorous proof, and identify truth with the existence of such a proof.

The precise analysis of this rather vague definition requires preliminary consideration of some basic concepts. The first of these concepts is the notion of formal system.[12] This is a refinement of the naive notion of a postulate system, and it will be convenient to introduce the new notion by taking an example of the latter notion and formalizing it. The example selected is Dickson's postulate set for group theory,[13] as follows:

[12] This statement is meant in the sense that the notion of formal system is taken as basic in the present paper, not that it is characteristic of formalism as such. Cf. below, §5.

[13] L. E. Dickson, *Definition of a group and a field by independent postulates*, Transactions of this Society, vol. 6 (1905), pp. 198–204.

DICKSON'S POSTULATES FOR GROUPS

A. *Primitive ideas.* $G, \circ$.

B. *Postulates.*

1. If a and b are elements of G, then $a \circ b$ is a uniquely determined element of G.
2. If a, b, c are any elements of G, then
$$a \circ (b \circ c) = (a \circ b) \circ c.$$
3. There exists an element e of G, such that for any element a of G
$$a \circ e = a.$$
4. For each element a of G there exists an element a' such that
$$a \circ a' = e.$$

This set of postulates determines a set of theorems which are derived from them by the processes of logical deduction. But, as we have seen in the discussion of logicism, this is not an objective characterization. It is necessary to specify in detail the way in which theorems are to be derived.

A *formal system* may be naively described as what such a postulate system becomes when we put in these refinements. More precisely, such a system is defined by a set of conventions, which I shall call its *primitive frame*. These conventions are of three kinds, as follows: first, those which specify the objects of the system, which I shall call its *terms*—in the case of group theory these are the elements of the group; second, those determining a set of propositions, which I shall call the *elementary propositions*, concerning these terms—in the case of group theory these are the equations; and third, those which specify which of the elementary propositions are true, that is, are theorems.[14] These specifications have the form of recursive definitions. Thus the term-specifications give a list of primitive terms, or *tokens*, together with *operations* and *rules of term formation* which describe how further terms are to be constructed from the tokens. The specifications for elementary propositions consist of a list of *predicates* (properties, relations, and so on) together with *rules of proposition formation* for constructing elementary propositions by means of them. Finally, the specifications for theorems consist of a set of elementary propositions, called *axioms*, which are stated outright to be true; together with *rules of deduction* showing how further theorems are to be derived.

[14] Throughout this paper I shall use 'theorem' in the sense of 'true proposition.' Thus an axiom is a theorem by definition.

As in recursive definitions generally a property of closure is understood, namely, that the specifications give all the entities recursively —for example, *all* the terms are obtained by the processes described in the term specifications, and so on.

For a system G, related to group theory, the primitive frame is as follows:

SYSTEM G

I. TERMS.

A. *Primitive terms, or tokens.* $e, a_1, \cdots, a_n$.

B. *Operations.* $\circ$ (binary), $'$ (unary).

C. *Rules of term formation.*

1. If A and B are terms, $A \circ B$ is a term.
2. If A is a term, A' is a term.

II. ELEMENTARY PROPOSITIONS.

A. *Predicates.* $=$ (binary).

B. *Rule.* If A and B are terms, then $A = B$ is an elementary proposition.[15]

III. THEOREMS.

A. *Axioms.* If A, B, C are terms, then—

1. $A \circ (B \circ C) = (A \circ B) \circ C$,
2. $A \circ e = A$,
3. $A \circ A' = e$.

B. *Rules of deduction.* For any terms A, B, C—

1. If $A = B$, then $B = A$.
2. If $A = B$ and $B = C$, then $A = C$.
3. If $A = B$, then $C \circ A = C \circ B$.
4. If $A = B$, then $A \circ C = B \circ C$.
5. If $A = B$, then $A' = B'$.

The following remarks are intended to clear up certain points in regard to the primitive frame. (What a formal system essentially is will concern us later; for the present we are interested only in the specifications by which a formal system is defined.)

First, as to the symbols used. In the above statement of the primitive frame the symbols 'e,' 'a_1,' $\cdots$, 'a_n,' '$\circ$,' '$'$,' '$=$' are used to designate definite constituents of the system; on the other hand the capital letters, parentheses, and also words such as 'proposition' are to be taken as part of ordinary language.[16] The capital letters, in

[15] On the omission of quotation marks here see Footnote 20.

[16] Ordinary language must here be understood as including the use of variables and parentheses in the way in which they are employed in mathematics.

particular, are intuitive variables whose values are unspecified terms. Thus the specifications III A are to be understood in the sense that for each particular determination of 'A,' 'B,' 'C,' each of the three propositions stated is an axiom; there are, therefore, infinitely many axioms subsumed under three *axiom-schemes*.[17]

Again, the conventions I, II, have a different character from III, and it is convenient to have a name for them collectively. I shall call these specifications and considerations based on them *morphological*, as opposed to the conventions III which will be called *theoretical*. The morphology of the system G has evidently a quite trivial character; in more sophisticated systems we may have a much more complicated morphology—involving different categories of terms, or even a whole hierarchy of types, relations between terms and definitions of complicated derived notions such as substitution. But we must suppose in all cases that the morphological specifications enable us to determine whether a given combination of symbols denotes a term (or an elementary proposition).[18]

These notions may be made somewhat clearer if I add a word or two concerning the interpretation. In the system G the $a_1, \cdots, a_n$ may be thought of as the generators of the group. On the other hand it may be shown that if in any theorem involving any a_i we substitute

[17] This term is due to J. von Neumann, *Zur Hilbertschen Beweistheorie*, Mathematische Zeitschrift, vol. 26 (1927), pp. 1–46.

[18] It is to some extent arbitrary what considerations are taken as morphological; for in case of systems with a complicated morphology—say with different types of terms—we can take these types of terms as new predicates and transfer the rules concerning them into the theory. If this is carried to the limit we should arrive at a formal system, like the system G, in which there is only one category of terms, and it is not really necessary to state any morphological rules beyond giving the number of arguments for each operation and predicate. The natural formulation of the primitive frame on such a basis would be—

I. Primitive ideas.
  A. Tokens (primitive terms).
  B. Operations (classified as to number of arguments).
  C. Predicates (do.).
II. Primitive theorems.
  A. Axioms (that is, elementary primitive theorems).
  B. Rules.

Such a system let us call a *completely formal system*. Evidently it is the more definite concept, and for a primary standard of mathematical rigor it forms the proper basis. G is an example of such a system. In most theories, however, there is a class of theorems, of a relatively trivial nature (from the point of view of that theory) which it is convenient to take unformalized. The concept of a formal system, with the possibility of some complication in the morphology, is thus more easily fitted to existing theories than that of a completely formal one.

an arbitrary term A for a_i, the result is again an elementary theorem. It follows that the a_i may also be thought of as variables, and that the system contains those general equations of group theory which do not contain more than n variables. To include all equations we must, naturally, postulate infinitely many primitive terms. If a substitution rule is added to the rules as Rule 6, then the axiom schemes may be replaced by three single axioms as follows:

1. $a_1 \circ (a_2 \circ a_3) = (a_1 \circ a_2) \circ a_3$,
2. $a_1 \circ e = a_1$,
3. $a_1 \circ a_1' = e$.

The above formal system has of course a quite trivial character. (I do not claim that it is an adequate formalization of all of group theory.) On account of limitations of time, however, it will have to do as an example.

4. **Discussion of the nature of a formal system.** Let us now turn to the discussion of some points about the nature of a formal system. We shall inquire in what sense we may think of a formal system as an object, and whether we are justified in the use of the words 'proposition' and 'true' in connection with it. Also, on account of the prominence given to the question by Hilbert, it is necessary to say something about the relation of a formal system to symbolism; in this section and the next I shall defend the thesis that the extreme nominalism, with which formalism is now popularly associated, is no essential part of the doctrine.

As a preliminary to this discussion we must take account of the distinction between the use of a symbol as denoting something and the consideration of that symbol as an object. This is illustrated by the two sentences[19]

1. ω is an order type.
2. ω is a Greek letter.

From these it would be false to conclude that a Greek letter is an order type. Of course, under certain circumstances it is legitimate to use a symbol to designate itself. But, for maximum definiteness, I shall, when talking about a symbol (or expression), use a specimen of that symbol (or expression) enclosed in single quotation marks as a name for it.[20]

[19] These are due to R. Carnap, *Logical Syntax of Language*, London and New York, 1937 (German edition, 1934), p. 156. Some changes in wording have been made.

[20] This notation is due to Frege (see, e.g., *Grundgesetze der Arithmetik*, vol. 1, Jena, 1893, p. 4) and is now quite generally used by logicians. However, the distinction itself is made in some manner by linguists as a matter of course (and doubtless was before Frege's day). The distinction is important wherever the discussion has

In setting up the primitive frame of a formal system we are, of course, introducing certain symbols; and, although we do not define these symbols in the ordinary sense, yet we do specify how they are to be used in connection with words of ordinary language (such as 'proposition' and 'true'). Thus, for G, in I A we say in effect that 'e,' 'a_1,' $\cdots$, 'a_n' are nouns; we do not say what they are the names of, but they are names of some objects or other, which objects we call terms. Let us call these symbols the *primitive term nouns*. Then in I C we state that certain further expressions, formed from the primitive term nouns, the *operators* '$\circ$' and '$'$,' and parentheses, are also term nouns. II says that '$=$' is a verb, and that elementary sentences are formed by placing it between two term-nouns. It is evident that

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specific reference to the relation between symbols and their referents; but I think the confusion caused by non-observance of the distinction in ordinary mathematics is somewhat exaggerated. For discussion of this question cf. R. Carnap, *Logical Syntax*, pp. 156–160, or W. V. Quine, *Mathematical Logic*, New York, 1940, §4.

The discussion in the text has primary reference to symbols and expressions which are nouns. With respect to other types of expressions there should be at least a threefold, rather than a twofold, distinction. Thus, in connection with sentences we need to distinguish (a) the sentence as asserting something, (b) a noun naming the proposition asserted, and (c) a name for the sentence as a linguistic phenomenon. In the present paper I use the same expression in senses (a) and (b), relying upon the context to make the distinction clear. There is abundant precedent for this uage, both in ordinary and technical discourse, and it avoids overburdening the discussion with excessive symbolism. Nevertheless, a systematic method of making the distinction would be desirable. Such a method would be to use "corners," as in Quine's "quasi-quotation" (loc. cit., pp. 33–37, or Journal of Symbolic Logic, vol. 2, p. 146). This should be subject to the same rules as his quasi-quotation in regard to substitution for intuitive variables, so that, for example, we might write the conventions II of the primitive frame for G as—

There is one binary predicate, $\ulcorner = \urcorner$.
If A and B are terms, then $\ulcorner A = B \urcorner$ is an elementary proposition.

If this is coupled with an understanding of what symbols are intuitive variables we should have a satisfactory notation.

The distinction between (a) and (b) may be what the Principia Mathematica (and also Frege) had in mind in introducing the assertion sign '$\vdash$.' However their explanations in this connection are somewhat vague. All that is clear is that '$\vdash$' is placed before a noun to make a sentence—that is, it is an intransitive verb. It is therefore natural to use this symbol for the single unary predicate in formal systems which have such; it then corresponds to Hilbert's 'ist beweisbar.'

The neglect of the distinction between (b) and (c) seems to me to cause as much confusion as the neglect of that between (a) and (c). On this point compare the discussion of 'presentation' below.

It should be understood that single quotation marks, and these only, are used in the technical sense stated. Double quotation marks are used as in ordinary language.

what has been said could be expanded into a way of stating the primitive frame, and that the new way of statement would be essentially equivalent to the preceding one.

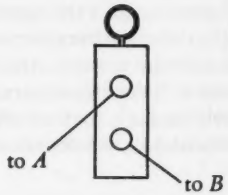
The new form of statement, however—aside from the disadvantages inherent in its greater cumbersomeness—is a decided misplacement of the emphasis. To a mathematician the older form of statement indicates that the particular symbols used are irrelevant; whereas the newer form appears to be tied to a particular choice of them. Let us call a particular statement of the primitive frame, with a particular choice of symbols, a *presentation* of the system. Then the newer mode of statement connotes that we have to do with a particular presentation. On the other hand, a formal system is to be understood as an abstraction from its presentations. This does not mean that a formal system is a mystic entity subsisting apart from its presentations. Indeed the question of the philosophic nature of such an abstraction is irrelevant to the present discussion. All we need to know is that when we think of a presentation as that of a formal system we regard certain features as essential and others as irrelevant, and that what we have to say applies equally well to any other presentation agreeing in the essential features. A similar remark applies to the words 'proposition,' 'predicate,' and so on, as opposed to the corresponding linguistic words 'sentence,' 'verb' (or 'sentential function'), and so on.[21]

The next question is the indeterminacy which we have already noticed in regard to the terms. The primitive frame does not assign any definite objects to the term nouns. We are then free to make what determinations we please. Let us call any determination of such objects, which assigns a unique term-object to each term noun and conversely, a *representation* of the system. Then, given any presentation of the system, we can always supply a representation. In fact we can construct a mechanical model in which the tokens are different kinds of buttons and the operations are different ways of tying them together. A scheme of doing this for the system G using auxiliary gadgets for the operations is shown by the diagrams. (Here it is to be understood that the holes are for the attachment of strings from the constituents A, B; while the rings are for attachment of a string connect-

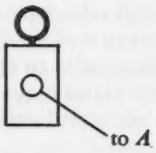
[21] It is not even necessary that the essential features be precisely formulated. As ordinarily used 'proposition' is a vague word, in that no exact criteria are given for deciding when two sentences express the same proposition. This vagueness is harmless because we seldom or never have occasion to ask the question indicated. The word is useful because it indicates in a rough way the level of abstraction at which we are operating; which is better than no indication at all, or a misleading one.

ing to an operation when the term $-A \circ B$ or A' is itself a constituent of a more complex term.)

When this is done the term nouns have been given a contensive meaning. Now the system does not admit of any further determination of such meaning. The indeterminacy which was noted for the terms does not extend to the predicates; for the rules II enumerate all of the



$A \circ B$



A'

significant instances of these predicates and III all the true instances, so that the predicates are defined completely by the primitive frame. Thus in the above representation the pairs of artifacts between which the relation of equality holds are defined by the primitive frame and nothing else. Consequently although the terms are unspecified, yet for any formal system it is always possible to construct a representation supplying this deficiency. The elementary theorems hold without regard to how this is done. The representation is therefore an accident; and a formal system is an abstraction from its representations just as it is from its presentations.

The notion of representation just discussed must be distinguished from another notion, that of *interpretation*. We speak of an interpretation of a formal system when the elementary propositions are put into relation with certain contensive propositions independently defined. Thus we can interpret G in a well known manner as a group of transformations, where two transformations are regarded as equal when they associate the same image with each point. This interpretation is a *valid* one, because the transformation processes associated with (formally) equal terms are also equal contensively; but the converse is not true. In the case of a representation such questions of validity cannot arise; all representations are isomorphic by definition.

A formal system, then, is something which is abstract in two ways; first, as to the symbols used to present it, and second, as to the objects used to represent it. The truth of the elementary propositions is in-

dependent of these contingencies. Moreover by virtue of the conventions III (and the closure property mentioned at the beginning) an elementary proposition is true when and only when there exists a sequence of elementary propositions, of which the given proposition is the last, such that every proposition in the sequence is either an axiom or is derived from certain of its predecessors by a rule of deduction. What an elementary proposition asserts is, therefore, precisely the existence of such a proof. Assuming that the specifications for axioms and rules have a sufficiently definite character—which is to be taken as part of the definition of a formal system—the checking of an alleged proof is an objective process.[22] An elementary proposition therefore states a question or problem such that an affirmative answer can be verified objectively, without any indeterminacy whatever.

5. **The notion of a calculus.** Although the notion of formal system is here taken as fundamental, yet other related notions are now more popular among specialists. As a typical such notion let us discuss here the notion of *calculus* as defined by Scholz.[23]

In a calculus it is explicitly stated that the objects we are dealing with are symbols. We start with a certain stock of symbols and with two kinds of rules for manipulating them. The first kind of rules, called *formation rules*, specifies recursively a certain set of *expressions* —that is, linear sets of symbols—which set I shall call *formulas*. The second set of rules, called *transformation rules*, specifies a class of formulas which I shall call *assertible formulas*; the rules consist of first a definite list of formulas, called here *axiom formulas*, which are assertible, and second, rules determining recursively how further assertible formulas are to be constructed.

Now it is evident that the rules of a calculus must be stated in a language such as English; and since the subject matter is symbols and expressions, the rules have somewhat the same character as the rules of syntax in grammar. For that reason the language is called the *syntax language*. On the other hand the expressions being talked about have also, in virtue of their being ordered by the rules, some of the properties of a language; and it is customary to refer to them as con-

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[22] We may require of a proof that not only is the sequence of propositions stated, but the reasons for the inference at each stage.

[23] H. Scholz, *Was ist ein Kalkül und was hat Frege für eine pünktliche Beantwortung dieser Frage geleistet*, Semester Berichte (Münster), 7th Semester, Summer, 1935, pp. 16–22. The notion is similar to that of R. Carnap, *Logical Syntax*, pp. 4–8, 167 ff., but is somewhat more explicit.

stituting the *object language*. There are thus two languages associated with a calculus, and the calculus may be thought of as a statement of the syntax of the object language in the syntax language. However the object language is a language only in a highly technical sense; it may or may not be a language in the usual sense.

Let us now consider the relations between a calculus and a formal system. We shall see that a formal system can be converted into a calculus, or something essentially similar, in two ways, either of which can be carried out mechanically. Conversely, by passing to a higher level of abstraction, a calculus can be converted into a formal system.

The process indicated in the discussion of presentation gives a method of forming a calculus from the symbols of the presentation. Thus, for the above presentation of the system G, take as symbols 'e,' 'a_1,' $\cdots$, 'a_n,' '$\circ$,' '$'$,' '$=$,' '$($,' '$)$'; take as formulas the elementary sentences, and as assertible formulas those which express true propositions. Then, reinterpreting the rules, we have a calculus. This process I shall call *calculization* of the presentation. It is subject to the objections which were mentioned in the discussion of presentation.

Another way of reducing a formal system to a calculus is to represent the system symbolically. A uniform process for doing this is illustrated by the following procedure for G; Let e, a_1, $\cdots$, a_n be respectively 'ϵ,' 'α_1,' $\cdots$, 'α_n,' and let 'ϕ,' 'ψ,' be two new symbols; for given terms A and B, let $A \circ B$ be the expression got by writing in order first 'ϕ,' then A, then B; and let A' be obtained by prefixing 'ψ' to A. That this is a representation follows by certain results of Łukasiewicz.[24] Then the terms will be a certain class of expressions which we can take as formulas. The resulting representation is not a calculus, because a relation between formulas and not a class of formulas is defined, but it has the essential nature of a calculus, and may be called a *generalized calculus*.[25] The reduction would lead precisely to a calculus if the original system had a single unary predicate; and by a rather trivial change every formal system can be reduced to one of that character.[26] By this mode of reduction—as in any representation—the symbols of the primitive frame belong to the syntax

[24] J. Łukasiewicz and A. Tarski, *Untersuchungen über den Aussagenkalkul*, Comptes Rendus de la Societé des Sciences et des Lettres de Varsovie, class III, vol. 23 (1930), p. 31. where references to earlier papers by Łukasiewicz (in Polish) may be found.

[25] The Church theory of conversion is an example of such a generalized calculus which occurs in the literature. See A. Church, *Mathematical Logic*, Princeton University (mimeographed), 1936, chap. 1.

[26] For the system G this can be done by introducing a new binary operation * and unary predicate $\vdash$, then replacing $A = B$ by $\vdash (A * B)$.

language, and the object language is the invented representation.[27]

In a calculus we do not consider all properties of the symbols, but only such as follow from the syntactical rules. Thus a calculus is to a certain extent abstract. We can make this abstractness explicit by converting to a formal system. In fact, using methods due to Tarski and Hermes[28] we can find a (represented) formal system such that the tokens are the object symbols, the terms are the expressions, there is a single unary predicate, and the elementary theorems ascribe this predicate to those and only those expressions which are assertible formulas. We then have a formal system of a special kind with a symbolic representation. The object language can now be abolished altogether. Other processes of formalization—in which the terms are formulas or other special classes of expressions—may be possible; and if the calculus is obtained by calculization of a formal system, a process of formalization can be found which will bring us back to another presentation of the original formal system.

These considerations show that a formal system and a calculus are essentially equivalent notions. A calculus is, in fact, merely a formal system tied to a special representation. This representation is accidental. A formal system is just as exact a notion; it does not force emphasis on extraneous considerations; it requires only one set of symbols, and these are adjuncts to the language we use, not that which we talk about; it leaves the way open for representations in terms of some subject matter more suggestive than symbols; and a symbolic representation can be manufactured for it when needed. For these reasons the notion of formal system is here preferred as the fundamental notion of formalism.

So much for the notions of formal system and calculus. The important point is that for elementary propositions we have an objective criterion of rigor. We turn now to the consideration of propositions which are not elementary.

6. **Metatheory.** In the actual study of a formal system we do not confine ourselves to deducing elementary propositions step by step. Rather we take the formal system as datum, and, having defined the

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[27] In the case of a completely formal system (see Footnote 16) with one unary predicate, this object language would consist of formulas and coordinating conjunctions only.

[28] E.g., see A. Tarski, *Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und der ω-Vollständigkeit*, Monatshefte für Mathematik und Physik, vol. 40 (1930), pp. 97–112, and H. Hermes, *Semiotik*, Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, new series, vol. 5, 1938.

object of study by setting up the primitive frame, we investigate it by any means available. In so doing we may formulate further propositions. I shall call these non-elementary propositions *metapropositions* (when true, *metatheorems*), and the method of study which gives rise to them the metatheoretic method.

We have already had an example of a metatheorem, namely, the substitution rule for the system G. It will be instructive to consider this theorem a little more carefully. First we need a precise definition of what we mean by the result of substituting the term C for the token a_i in the term A. Let us call this result A^*. Ordinarily we define it with reference to the symbols used to designate A and C; but this apparent dependence on symbols can be avoided by defining A^* recursively thus (here '$\equiv$' denotes definitional identity):

1. $a_i^* \equiv C$;
2. If A is a token distinct from a_i, $A^* \equiv A$;
3. $(A \circ B)^* \equiv A^* \circ B^*$;
4. $(A')^* \equiv (A^*)'$.

The theorem, now, is the following: Suppose that A and B are terms such that

$$A = B, \tag{1}$$

and C is a term, then

$$A^* = B^*. \tag{2}$$

Without going into details we may notice that the proof contains two essential steps. First, the conclusion is true if the hypothesis is an axiom; in fact then the conclusion is an axiom. Next, suppose the hypothesis is derived by a rule of procedure from certain premises, and that we have already shown that the theorem holds for these premises; then the same rule of procedure allows us to derive the conclusion from the transforms of the premises. At this point it is customary to say that the theorem follows by induction. However, what we have actually done is to exhibit a process whereby, given a proof of (1), we can convert it step by step into a proof of (2).

The variety of metatheorems is immense. Some further examples are the following. In the first place we may have simple combinations of the elementary propositions, namely,

$$a_1 \circ e = e \circ a_1 = a_1,$$

which may be regarded as a conjunction of three elementary theorems. Then there are general theorems conjoining whole classes of

elementary propositions such as

$$A' \circ A = e$$

(the axiom schemes, considered as single propositions, are of this character); or derived rules, such as—

$$\text{If} \quad A \circ B = e, \quad \text{then} \quad B = A'.$$

Again we may introduce additional operations and predicates by recursive definitions, and consider general theorems concerning them, like the one just treated or the generalized associative law. Moreover there are properties of the system as a whole; such as consistency, completeness, and resolvability. We may also consider the relations of a system to other systems, in particular to its own subsystems and supersystems, its extensions of various kinds and so on. Here we tap the resources of modern algebra, with its extensions and ideals, including the four morph brothers—homo-, iso-, auto-, and holo-. Such metatheorems form the very life and soul of mathematics. Finally we may have considerations relating a system to extraneous (non-mathematical) considerations, idealistic hypotheses, or what not, such as the semantical investigations of Tarski and Carnap.

If the notion of formal system is taken as a basis, these metatheorems are expressed in the same language as the elementary theorems. Metatheory is thus not quite the same notion as Hilbert's metamathematics, although it is related to it. Hilbert sets up a calculus which may be regarded as the calculization of a portion of mathematics; and then calls the study of this calculus metamathematics. This is, to say the least, an unfortunate terminology. Hilbert, presumably, does not mean to imply that mathematics is merely a meaningless playing with the counters of his calculus, but rather that mathematics is an activity carried out in an interpreted object language from which meaning has been abstracted in the process of calculization; on the other hand his calculization does seem to imply, insofar as mathematics is formal, that the only legitimate mathematics is the explicit derivation of elementary theorems. But metatheorems have as good a right to be called mathematical as elementary ones; moreover from the general point of view which I shall mention later, Hilbert's metamathematics is a branch of mathematics rather than something which transcends it. Furthermore, we have already seen that if Hilbert's calculus is formalized we come back to another presentation of the original mathematics in the syntax language. The absolute separation of mathematics and metamathematics is fictitious. Hilbert's calculiza-

tion had, I think, a psychological purpose, which it served well; but there is no sound basis for affirming the necessity of any such process.

As to the criteria of rigor for metatheoretic proofs, it is evident that we cannot handle the question as definitely as for the elementary ones. Many metatheoretic propositions have a vague character and require analysis. Such analysis can be given only by considering the specific kinds of metapropositions concerned. However, supposing this analysis made, we can divide metatheoretic proofs into two kinds, constructive and non-constructive. A constructive proof is one which, like the above proof of the substitution theorem, exhibits a process which can actually be carried through in any particular case arising under the hypotheses. For such a proof the criterion of rigor is reduced in the last analysis to that for the elementary propositions, and it has the same objective character. On the other hand the non-constructive proof depends on idealistic or other extraneous assumptions; and therefore we cannot regard them as purely formal.

An especially important class of metatheorems is constituted by certain theorems of incompleteness which have played a prominent role in recent years. These started with the Löwenheim[29] theorem of 1915, which was later refined by Skolem.[30] The upshot of these theorems was that any system of a certain kind would have an enumerable model in a sense which cannot be gone into here; since these systems have also interpretations which are non-enumerable, it follows that the interpretation cannot be unique as to structure. This, of course, is a non-constructive theorem. Sixteen years later Gödel[31] proved constructively that in systems strong enough for the usual mathematical purposes there are elementary propositions such that, if the system is consistent, neither these propositions nor their formal negatives within the system can be proved. A similar theorem of Kleene and Rosser, published in 1935,[32] shows that a formal system cannot have

[29] L. Löwenheim, *Über Möglichkeiten im Relativkalkul*, Mathematische Annalen, vol. 76 (1915), pp. 447–470.

[30] Th. Skolem, *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen*, Videnskapsselskapet Skrifter, class I, 1920, no. 6.

[31] K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*, Monatshefte für Mathematik und Physik, vol. 38, pp. 173–198. For expositions of this famous theorem see also D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, vol. 2, 1939, pp. 269–289, also J. B. Rosser, *An informal exposition of proofs of Gödel's theorem and Church's theorem*, Journal of Symbolic Logic, vol. 4 (1939), pp. 53–60.

[32] S. C. Kleene and J. B. Rosser, *The inconsistency of certain formal logics*, Annals of Mathematics, (2), vol. 36 (1935), pp. 630–636. A revised proof of this theorem is con-

a kind of completeness as to term-formation and at the same time be capable of formalizing—in a sense which cannot be explained here—its own deductive processes.

7. **Definition of mathematics.** We are now in a position to discuss the formalist definition of mathematics. In this connection the incompleteness theorems just discussed have important consequences for the formalist conception of mathematics. Indeed they show that it is not feasible to consider mathematics as the development of a single formal system. There are other reasons, too, which support such a view; for the arbitrary nature of the definitions constituting the primitive frame of a formal system shows that, in principle at least, all formal systems stand on a par.[33] The essence of mathematics lies, therefore, not in any particular kind of formal system, but in formal structure as such. *Mathematics*, then, *should be defined as the science of formal systems in general*; it should include all propositions elementary or metatheoretic, relating to one system or several, or to formal systems in general, so long as their criteria of truth depend on formal considerations alone.

This last proviso would exclude from mathematics proper all theorems depending on non-constructive proofs. This does not mean that these theorems are without mathematical interest. They are simply mixtures of mathematics and something else. As the example of the Löwenheim-Skolem theorem shows, these theorems are often of great significance for us; moreover they may lead to purely mathematical propositions at a higher level of formalization.

Mathematics, so conceived, has the following characteristics. First, it includes everything we know as mathematics. Classical analysis, for example, although not constructive when regarded as a metatheory of arithmetic, nevertheless can be embodied in a more inclusive formal system.[34] Second, mathematics is a science, in that it consists of propositions—not formulas but real propositions, with a definite criterion of truth. This criterion of truth is capable of fully

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tained in my paper *The paradox of Kleene and Rosser* (not yet published), and a considerable simplification in a paper, under the same title as that of Kleene and Rosser, which is now in preparation.

[33] This statement is not to be confused with the naive view that mathematics consists of more or less autonomous postulate systems. For if we could formulate a completely adequate logic, then these postulates could be added to the hypotheses of the theorems, and so all the postulate systems could be subsumed under one master-system—logic. But the point is that we can not have any such logic.

[34] Such formal systems, insofar as they are known at present, are not wholly satisfactory. Cf. Church's appendix to F. Gonseth, *Philosophie Mathématique*, Paris, 1940.

objective verification or proof—the primary standard of rigor for such a proof being the definition of derivability of an elementary proposition. On the other hand the gist of mathematics is that we make definitions by recursion, and then draw particular consequences by applying the definitions and general consequences by mathematical induction. There is thus a certain amount of justice in the view that mathematical theorems are consequences of definitions; but since the definitions are recursive, mathematics does not have the trivial character which that seems to imply. Likewise there is justice in the view that mathematical induction is a process of cardinal importance. However, it is not a mysterious power of the human mind, as Poincaré seemed to think, but is simply a corollary of recursive definitions; for if we demonstrate by mathematical induction that all members of a recursively defined class have a certain property, that demonstration is constructively valid because *by definition* every member of the class can be reached by the inductive process.

At an earlier stage I stated that most of the positive criticisms of the intuitionists find their justification in modern formalism. We have seen that this is so for the Kronecker demand for constructivity; indeed we are more stringent in this particular than the intuitionists themselves.[35] In the preceding paragraph we have also seen that their emphasis on mathematical induction has some justification. As for the law of excluded middle, it is necessary first to explain what it means. If we interpret it as a metaproposition concerning a formal system S, to the effect that every elementary proposition of S is true or false, then we need to know the meaning of 'false' (not to mention 'or'). Now although we know what it means to verify an elementary proposition of S, yet we have no constructive definition of what it means to say it is false. We might define constructive falsity in any one of a number of different ways; but for any of the suggested definitions the law of excluded middle holds only if the system has a relatively trivial character.[36]

There is not time to go further with this. We can say, however, that it is useless to deny that intuition is involved in mathematics—if one defines intuition properly the statement is a tautology. However, we do not have to postulate a metaphysical character for it;

[35] This is shown by Gödel's reduction of classical arithmetic to that of intuitionism (*Zur intuitionistischen Arithmetik und Zahlentheorie*, Ergebnisse eines mathematischen Kolloquiums, vol. 4 (1932), pp. 34–38). The consistency of classical arithmetic cannot be proved *constructively* in any such simple manner.

[36] These matters I plan to discuss more fully in a forthcoming paper *Some properties of formal deducibility*.

indeed to many of us it seems that this intuition is an empirical, linguistic phenomenon. On the other hand if we subtract from intuitionism its metaphysics, the remaining differences between it and formalism are relatively superficial as regards mathematical rigor. The main point is that the intuitionist prefers to confine attention to certain special types of formal systems. This leads to considerations, going beyond the definition of rigor, to which we must now devote some attention.

8. **Acceptability.** It is obvious that we are not interested in all formal systems. Those considerations which lead us to choose one system rather than another need to be distinguished from those relating to the truth of propositions concerning a formal system once chosen. To make this distinction I shall call that property of a formal system which leads us to adopt it for consideration its *acceptability*. This is evidently a quasi-truth concept which applies to a system as a whole.

In strictness acceptability is irrelevant to the problem of mathematical rigor. A proof of a proposition relating to a formal system is rigorous if and only if it satisfies certain objective criteria which are independent of acceptability. But it is necessary to give a brief account of acceptability for two reasons: first, because there has been some confusion in regard to it, and second because it sheds some light on the significance of mathematical rigor from the point of view of science.

The first point is that acceptability is relative to a purpose. It usually means that we are interested in some interpretation of the formal system; and the validity of the interpretation in relation to the subject matter is the prime consideration. Without statement of the purpose any discussion of acceptability is futile. Moreover the various schools of thought are not necessarily in conflict with one another. There is no one absolutely acceptable system; for different purposes entirely different systems may be acceptable.

Again if the purpose is an empirical one, as it is in physics, the question of acceptability is empirical also. Neither intuitive evidence, which is stressed by the intuitionists, nor demonstrable consistency, which is insisted on by Hilbert, is of more than secondary importance. From this point of view I think we must agree that, at the present state of our knowledge, some system of classical analysis is acceptable, however meaningless it may be to the intuitionists, and however far we may be from a formulation with a consistency proof. Of course if an inconsistency should be found, we should have to modify the system; but at the present time the acceptability of analysis is an empirical fact which requires no justification.

It is perhaps worth mentioning that the demand for intuitive evidence and the demand for consistency proof amount, in the last analysis, to nearly the same thing. For a thoroughly satisfactory consistency proof would exhibit a process whereby, given a classical proof with an intuitionistically significant conclusion, we could transform it into an intuitionistic proof of the same conclusion. Ackermann's recent proof of the consistency of arithmetic[37] does just this for a certain class of propositions. The proof is accomplished by a series of total replacements ("Gesamtersetzungen"); if the conclusion of the original proof is a particular numerical formula, the last total replacement gives an intuitive proof of the same formula. A complete solution of the consistency problem for analysis would show that any classical proof of a proposition with a direct intuitive meaning could be transformed into a purely constructive proof. In view of the Löwenheim-Skolem theorem there is reasonable doubt as to the possibility of such a transformation. And even if we succeed in finding it, it is not improbable that we shall always have uses for systems whose consistency is unknown.

The upshot of this is that we should cultivate a tolerant attitude in matters of acceptability.[38] Acceptability is not a question of right and wrong but of choice of subject matter. Such a choice should be free; and some difference of opinion is not only allowable but desirable. As mathematicians we should know to what sorts of system our theorems—if formalized—belong; but to exclude systems which fail to satisfy this or that criterion of acceptability is pedantry.

From the point of view of formalism, then, the question of truth of a naive mathematical proposition can be split into two parts. The first is the discovery of the formal system to which the proposition is to be referred, and the proof of the corresponding formal proposition; this is the problem of mathematical rigor, and we have seen that it has an objective character. The second is the acceptability of that system for the purpose in hand; this is a problem of applied mathematics.

PENNSYLVANIA STATE COLLEGE

[37] W. Ackermann, *Zur Widerspruchsfreiheit der Zahlentheorie*, Mathematische Annalen, vol. 117 (1940), pp. 162–194.

[38] Cf. Carnap's "Principle of Tolerance" (*Logical Syntax*, p. 51 ff.); also Weyl, H., *Die heutige Erkenntnislage in der Mathematik*, Sonderdrucke des Symposion, vol. 3, Erlangen, 1926, pp. 31 ff.

SPACE CREMONA TRANSFORMATIONS OF ORDER $m+n-1$[1]

EDWIN J. PURCELL

1. **Introduction.** This paper discusses a space Cremona transformation of order $m+n-1$ (m, n any integers) generated by two rational twisted curves. One special position of the defining curves gives rise to an involution recently described,[2] while another special position results in an involution somewhat similar to one which was defined in a different manner by Montesano.[3]

2. **Cremona transformation.** Consider a curve C_n of order n having $n-1$ points on each of two skew lines d and d', and a curve C_m' of order m having $m-1$ points on each of d and d' (m, n, any integers). A generic point P determines a ray through it intersecting C_n once in α and d once in β. P also determines a ray through it intersecting C_m' once in γ and d once in δ. We define P', the correspondent of P, to be the intersection of lines $\alpha\delta$ and $\beta\gamma$.

It is to be noted that if C_n should become identical with C_m' but d and d' remain distinct, there would result the Cremona involution we discussed in a recent paper (loc. cit.).

Let the equations of d be $x_1=0$, $x_2=0$, and those of d' be $x_3=0$, $x_4=0$. Let C_n be

$$x_1 = (as + bt)\prod_1^{n-1}(t_i s - s_i t), \qquad x_2 = (cs + dt)\prod_1^{n-1}(t_i s - s_i t),$$

$$x_3 = (es + ft)\prod_n^{2n-2}(t_i s - s_i t), \qquad x_4 = (gs + ht)\prod_n^{2n-2}(t_i s - s_i t),$$

where s_i, t_i for $i=1, 2, \cdots, n-1$ are values of the parameters of C_n for points on d, and for $i=n, n+1, \cdots, 2n-2$, for points on d'.

Let the equations of C_m' be

$$x_1 = (AS + BT)\prod_1^{m-1}(T_i S - S_i T), \quad x_2 = (CS + DT)\prod_1^{m-1}(T_i S - S_i T),$$

$$x_3 = (ES + FT)\prod_m^{2m-2}(T_i S - S_i T), \quad x_4 = (GS + HT)\prod_m^{2m-2}(T_i S - S_i T),$$

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[1] Presented to the Society, September 10, 1940.

[2] E. J. Purcell, *A multiple null-correspondence and a space Cremona involution of order* $2n-1$, this Bulletin, vol. 46 (1940), pp. 339–444.

[3] D. Montesano, *Su una classe di trasformazioni involutorie dello spazio*, Rendiconti del' Istituto Lombardo di Scienze e Lettere, (2), vol. 21 (1888), pp. 688–690.

where S_i, T_i for $i=1, 2, \cdots, m-1$ are values of the parameters of C_m' for points on d, and for $i=m, m+1, \cdots, 2m-2$, for points on d'. Then the equations of the transformation are

$$x_1' = k(Q_1x_3 + Q_2x_4)\left(\prod_1^{n-1}\theta_i\right)\left(\prod_1^{m-1}\Phi_i\right),$$

$$x_2' = k(R_1x_3 + R_2x_4)\left(\prod_1^{n-1}\theta_i\right)\left(\prod_1^{m-1}\Phi_i\right),$$

$$x_3' = K'(r_2x_1 - q_2x_2)\left(\prod_n^{2n-2}\theta_i\right)\left(\prod_m^{2m-2}\Phi_i\right),$$

$$x_4' = K'(q_1x_2 - r_1x_1)\left(\prod_n^{2n-2}\theta_i\right)\left(\prod_m^{2m-2}\Phi_i\right),$$

where $k \equiv (bc-ad)$, $K' \equiv (FG-EH)$, and

$$Q_1 \equiv (AH - BG), \qquad Q_2 \equiv (BE - AF),$$
$$R_1 \equiv (CH - DG), \qquad R_2 \equiv (DE - CF),$$
$$q_1 \equiv (ah - bg), \qquad q_2 \equiv (be - af),$$
$$r_1 \equiv (ch - dg), \qquad r_2 \equiv (de - cf),$$
$$\theta_i \equiv \{t_i(bx_2 - dx_1) - s_i(cx_1 - ax_2)\},$$
$$\Phi_i \equiv \{T_i(Hx_3 - Fx_4) - S_i(Ex_4 - Gx_3)\}.$$

The inverse transformation is

$$x_1 = K(q_1x_3' + q_2x_4')\left(\prod_1^{n-1}\phi_i'\right)\left(\prod_1^{m-1}\Theta_i'\right),$$

$$x_2 = K(r_1x_3' + r_2x_4')\left(\prod_1^{n-1}\phi_i'\right)\left(\prod_1^{m-1}\Theta_i'\right),$$

$$x_3 = k'(R_2x_1' - Q_2x_2')\left(\prod_n^{2n-2}\phi_i'\right)\left(\prod_m^{2m-2}\Theta_i'\right),$$

$$x_4 = k'(Q_1x_2' - R_1x_1')\left(\prod_n^{2n-2}\phi_i'\right)\left(\prod_m^{2m-2}\Theta_i'\right),$$

where $K \equiv (BC-AD)$, $k' \equiv (fg-eh)$,

$$\phi_i' \equiv \{t_i(hx_3' - fx_4') - s_i(ex_4' - gx_3')\},$$
$$\Theta_i' \equiv \{T_i(Dx_1' - Bx_2') - S_i(Ax_2' - Cx_1')\}.$$

Both the direct and inverse transformations are of order $m+n-1$, where m and n are any integers.

The fundamental system and its images for the direct transformation are as follows.

d is an $(n-1)$-fold F-line of simple contact. The fixed tangent planes are $\theta_i=0$, where $i=1, 2, \cdots, n-1$. It is of the first species and its P-surface consists in the planes $\phi_i'=0$, where $i=1, 2, \cdots, n-1$, which pass through d'.

d' is an $(m-1)$-fold F-line of simple contact. The fixed tangent planes are $\Phi_i=0$, where $i=m, m+1, \cdots, 2m-2$. It is of the first species and its P-surface consists in the $m-1$ planes $\Theta_i'=0$ through d, where $i=m, m+1, \cdots, 2m-2$.

Each of the $m-1$ intersections of C_m' and d is an n-fold isolated F-point. Their P-surfaces are $\Theta_i'=0$, where $i=1, 2, \cdots, m-1$, respectively.

Each of the $n-1$ intersections of C_n and d' is an m-fold isolated F-point. Their P-surfaces are $\phi_i'=0$ $(i=n, n+1, \cdots, 2n-2)$ respectively.

The $(n-1)(m-1)$ lines of intersection of the $n-1$ fixed tangent planes through d with the $m-1$ fixed tangent planes through d' are simple F-lines without contact. They are of the second species.

The $(m-1)(n-1)$ lines joining the $m-1$ n-fold isolated F-points on d with the $n-1$ m-fold isolated F-points on d' are simple F-lines without contact. They are of the second species.

We may obtain a description of the fundamental system of the inverse transformation by interchanging m and n, C_n and C_m', θ_i and Θ_i', Φ_i and ϕ_i', wherever they appear in the foregoing.

C_n, d, and d' lie on the same quadric surface Q, and C_m', d, and d' lie on a quadric surface Q'. These quadrics may be the same or distinct and, while this does not affect the preceding discussion, the invariant systems for the two cases are different.

When Q and Q' are distinct, they intersect in d, d', and two transversals l_1 and l_2. The d and d' are common generators of the μ-systems of the two quadrics, while l_1 and l_2 are common generators of their λ-systems. The transformation sends each λ-generator of Q over into a λ-generator of Q', and each λ-generator of Q' over into a λ-generator of Q. Thus Q as a whole corresponds to Q' and vice versa. Each λ-generator of either quadric belongs to a cycle of index four—that is, four applications of the transformation leave every λ-generator invariant. The transformation interchanges C_n and C_m'. The points of l_1 are in involution; thus l_1 is an invariant line and the two fixed points of the involution are invariant points. Similarly for l_2. These four invariant points are the only invariant points that are not also F-points.

Let us now consider the case where C_n, C_m', d, and d' all lie on the

same quadric $Q \equiv x_1x_4 - x_2x_3 = 0$. The transformation causes C_n and C_m' to interchange. The pencil of planes $x_4 - \lambda x_3 = 0$ is in involution with the pencil $x_2 - \lambda x_1 = 0$ and this makes each λ-generator of Q invariant. Consequently Q is invariant. The locus of invariant points is a curve K_{m+n} of order $m+n$ lying on Q. K_{m+n} passes through the $m+n-2$ points of intersection of C_n and C_m' and intersects d and d' in the $m+n-2$ isolated F-points on each of them. It intersects every μ-generator of Q in $m+n-2$ points and intersects every λ-generator in two points.

3. **Involution.** Consider a twisted curve C_n having $n-1$ points Δ_i on a straight line d, and a curve C_m' having $m-1$ points Σ_i on the same straight line d (m, n any integers). A generic point P determines a ray through it intersecting C_n in α and d in β, and also a ray through it intersecting C_m' in γ and d in δ. We define P', the correspondent of P in the involution, to be the intersection of lines $\alpha\delta$ and $\beta\gamma$.

If, in §2, we make d and d' identical, we obtain an involution of this kind. However, the curves C_n and C_m' of the present section do not necessarily lie on quadric surfaces.

The involution is of order $m+n-1$.

The fundamental system and its principal images follow.

d is an $(n+m-2)$-fold F-line of simple contact. The fixed tangent planes are $\theta_i = 0$, where $i = 1, 2, \cdots, n-1$, and $\Theta_i = 0$, where $i = 1, 2, \cdots, m-1$. It is of the second species and counts $(n+m-1)\cdot(n+m-2)$ times in the intersection of any two homaloids.

Points Δ_i are isolated F-points. Their P-surfaces are the planes $\theta_i = 0$ $(i = 1, 2, \cdots, n-1)$ respectively.

Points Σ_i are isolated F-points. The P-element of each is $\Theta_i = 0$ $(i = 1, 2, \cdots, m-1)$ respectively.

As we have seen, a general point P determines with d a plane π intersecting C_n in α and C_m' in γ. Call L the intersection of lines $\alpha\gamma$ and d. Then J, the harmonic conjugate of L with respect to α and γ, will be the only invariant point of π other than points of d. As π makes one revolution about d, α moves in π and crosses d $n-1$ times; γ also moves in π, crossing d $m-1$ times. As α approaches d, J approaches the same point on d, and the locus of J intersects d in all the points d has in common with C_n and C_m'. The locus of J is a rational curve K_{m+n-1} of order $m+n-1$ having $m+n-2$ points on d. K_{m+n-1} is the locus of invariant points.

It is clear that the line PP' intersects K_{m+n-1} in J and d in L, and that P and P' are harmonic conjugates with respect to[4] J and L.

[4] Compare with Montesano, loc. cit.

4. **Lower order for particular positions of the defining elements.** Each of the fixed tangent planes $\theta_i=0$ mentioned in the contact conditions for the involution passes through d and is tangent to C_n at the corresponding Δ_i. The fixed tangent planes $\Theta_i=0$ are similarly related to the curve C_m'.

If C_n and C_m' are so situated that a plane of $\theta_i=0$ $(i=1, 2, \cdots, n-1)$ coincides with a plane of $\Theta_i=0$ $(i=1, 2, \cdots, m-1)$, the order of the involution is reduced by one. In this way we may reduce the order by any integer up to, and including, the smaller of the two numbers $n-1$ and $m-1$.

UNIVERSITY OF ARIZONA

NOTE ON AUTOPOLAR CURVES[1]

MALCOLM FOSTER

1. **Introduction.** The aim of this paper is to study those curves which are autopolar with respect to the parabola $2\eta = \xi^2$. The method, which is believed to be new, is to consider these curves as special solutions of those differential equations which are invariant under the dual substitutions for the above conic of reference.[2] It will be obvious that this method may be readily modified for the study of curves which are autopolar with respect to any conic. The parabola $2\eta = \xi^2$ has been chosen for the sake of the simplicity of the substitutions.

2. **Dual substitutions for the conic of reference.** In the ordinary differential equation,

$$f(x, y, y', y'', \cdots, y^{(n)}) = 0, \tag{1}$$

let us make the well known dual substitutions[3]

$$\begin{aligned} &x = P = Y', \qquad y = Y'X - Y, \qquad p = y' = X, \\ &y'' = 1/Y'', \; y''' = -Y'''/Y''^3, \cdots \end{aligned} \tag{2}$$

for which the conic of reference is the parabola $2\eta = \xi^2$. We obtain a new differential equation,

$$f(Y', Y'X - Y, X, 1/Y'', \cdots) = 0, \tag{3}$$

whose solution is, let us say,

$$\phi(X, Y, c_1, c_2, \cdots, c_n) = 0. \tag{4}$$

If we eliminate X, Y from equation (4) and the following two equations,[4]

$$x\frac{\partial \phi}{\partial Y} + \frac{\partial \phi}{\partial X} = 0, \qquad -y\frac{\partial \phi}{\partial Y} = Y\frac{\partial \phi}{\partial Y} + X\frac{\partial \phi}{\partial X}, \tag{5}$$

we shall have the solution of the original differential equation (1), which we shall denote by

$$F(x, y, c_1, c_2, \cdots, c_n) = 0. \tag{6}$$

3. **Geometrical interpretation.** Let C be any curve of the family

[1] Presented to the Society, October 28, 1939.

[2] A. R. Forsyth, *A Treatise on Differential Equations*, 3d edition, 1903, pp. 45–47.

[3] Forsyth, op. cit.

[4] Forsyth, loc. cit.

(6). As a point P traverses C, the polar of P relative to the conic of reference will envelop some member C' of the family (4), and vice versa. That is, C and C' are polar reciprocals.

If (1) is of the first order, equations (1) and (3) may have singular solutions. If E and E' denote the envelopes of the families (4) and (6), it is evident from the above that E and E' are also polar reciprocals.

In addition to these relations between the families (4) and (6), there are several well known relations between the extraneous loci which may exist in connection with the integral curves. For example, a cuspidal locus for the integral curves (4), $[(6)]$, will correspond to the locus of points of inflexion for the integral curves (6), $[(4)]$.[5]

4. **Condition that $y=f(x)$ be autopolar.** If a curve C, $y=f(x)$, be autopolar, that is, its own polar reciprocal, the polar of any point (x_1, y_1) on C will be tangent to the curve at some other point (x_2, y_2). Relative to the above conic of reference, the polar of any point (x, y) on C is $x\xi-\eta-y=0$, which we may write

$$x\xi - \eta - f(x) = 0. \tag{7}$$

This is a one-parameter family of lines with x as the parameter, and their envelope will be found by the elimination of x from (7) and the following equation,

$$\xi - f'(x) = 0. \tag{8}$$

The necessary and sufficient condition that C be autopolar is that on eliminating x we shall get $\eta=f(\xi)$ as the envelope. From (7) and (8) we have $xf'(x)-\eta-f(x)=0$; and on replacing η by $f(\xi)$, or $f[f'(x)]$, we have

$$f[f'(x)] + f(x) - xf'(x) = 0. \tag{9}$$

We have, therefore, the following theorem:

THEOREM 1. *A necessary and sufficient condition that a curve $y=f(x)$ be autopolar with respect to the conic $2\eta=\xi^2$ is that the relation* (9) *be satisfied.*

It is interesting to note that (9) is of the Clairaut type.

5. **Relations between conjugate pairs, (x_1, y_1) and (x_2, y_2).** Consider any point $P_1(x_1, y_1)$ on an autopolar curve C. The polar of P_1 is $x_1\xi-\eta-y_1=0$; and since this line is tangent to C at some point

[5] Sophus Lie and Georg Scheffers, *Geometrie der Berührungstransformationen*, vol. 1, 1896, pp. 24–27.

$P_2(x_2, y_2)$, the equation of the polar of P_1 must be satisfied by the coördinates of P_2. Consequently,[6]

$$y_1 + y_2 = x_1x_2. \tag{10}$$

From the theory of polar reciprocals we know that the polar of P_2 is tangent to C at P_1, and since the slopes of the polars of P_1 and P_2 are respectively x_1 and x_2, we have

$$x_1 = f'(x_2), \qquad x_2 = f'(x_1). \tag{11}$$

Let us call such a pair of points, P_1 and P_2, a conjugate pair, and say that each is the conjugate of the other. The relations (11) are also evident from the set of substitutions (2) in which $x=P$, $X=p$.

From (10) and (11) we have the following theorem:

THEOREM 2. *On any autopolar curve with respect to the conic* $2\eta=\xi^2$, *the product of the slopes at a conjugate pair is equal to the sum of the ordinates at these points.*

6. **Self-conjugate points.** For any real conjugate pair, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, let us suppose $x_2>x_1$. Let us also assume that $f(x)$ is defined for all values of x between x_1 and x_2. Now let x_2 decrease, that is, let P_2 approach the original position of P_1. When P_2 arrives at this position. P_1 will have arrived at the original position of P_2. Hence x_1 must increase as x_2 decreases. Consequently, between each real conjugate pair there exists a point $P_3(x_3, y_3)$ which is self-conjugate. From (10) we shall have $2y_3=x_3^2$, and therefore P_3 lies on the conic of reference. Since the polar of P_3 is tangent to the conic of reference at this point, we see that every autopolar curve has a common tangent with this conic.

Also from (11), and the law of the mean, we have

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} = -1 = f''(x_4), \qquad x_1 < x_4 < x_2, \tag{12}$$

and since this relation is satisfied by every conjugate pair, no matter how near they may be to P_3, we see that

$$f''(x_3) = -1. \tag{13}$$

From (11) and (13) we readily find the curvature of any autopolar curve at P_3 to be

$$K = \frac{-1}{[1 + f'^2(x_3)]^{3/2}} = \frac{-1}{(1 + x_3^2)^{3/2}}.$$

[6] It is obvious from (2) that (10) is simply a restatement of (9).

It is readily verified that, except for sign, this is the curvature of the conic of reference at P_3. Hence we have:

THEOREM 3. *For every autopolar curve with respect to the conic* $2\eta = \xi^2$, *the curvature at the self-conjugate points* P_3 *is the same, except for sign, as the curvature of the conic at* P_3.

We note that this theorem applies also to the case of the general conic by virtue of the fact that under any collineation the ratio of the curvatures of two curves at a point at which they are tangent is invariant.[7]

From (12) it also follows that between real conjugate pairs $f'(x)$ is a decreasing function.

7. **Loci associated with a conjugate pair.** Consider the locus of Q, the intersection of the polars for any conjugate pair, P_1 and P_2. The coördinates (ξ, η) of Q are readily found from the equations of the polars of P_1 and P_2; we find

$$\xi = \frac{y_1 - y_2}{x_1 - x_2}, \qquad \eta = \frac{x_2 y_1 - x_1 y_2}{x_1 - x_2}. \tag{14}$$

By means of (11) these equations of the locus of Q are readily given in terms of one parameter, say x_1.

Since the polar of Q is the line P_1P_2, we see that the locus of Q and the envelope of the lines P_1P_2 are also polar reciprocals.

It will be of interest to consider also the locus of S, the mid-point of the segment P_1P_2.

We shall also consider the locus of R, the intersection of normals at conjugate pairs. The coördinates of R are readily found to be

$$\xi = \frac{x_1^2 + x_1 x_2 y_1 - x_2^2 - x_1 x_2 y_2}{x_1 - x_2}, \quad \eta = \frac{x_2 + x_1 y_2 - x_1 - x_2 y_1}{x_1 - x_2}. \tag{15}$$

From §6 it is evident that the loci of Q and S must pass through P_3.

8. **Differential equations invariant under the above transformation.** Any differential equation of the form

$$f(x)y'' + f(y') = 0 \tag{16}$$

becomes, on using (2), $f(X)Y'' + f(Y') = 0$, which is identical with (16). Let us denote the solution of (16) by

[7] G. Fubini and E. Čech, *Introduction à la Géométrie Protective Différentielle des Surfaces*, 1931, pp. 17–20.

$$\phi(x, y, c_1, c_2) = 0. \tag{17}$$

As a point P traverses some member C of (17), the polar of P will, in general, envelop some other member C' of the same family. We wish to determine if there are any members of the two-parameter family (17), which are autopolar. The method for the determination of these curves will be illustrated by particular examples.

EXAMPLE 1. $(x+1)y''+y'+1=0$. The solution of this is

$$y = c_1 \log (x + 1) - x + c_2. \tag{18}$$

Here $f(x)=c_1 \log (x+1)-x+c_2$, and $f'(x)=c_1/(x+1)-1$. If we put these expressions in (9), we get $(c_1-c_1 \log c_1-1-2c_2)x+c_1-c_1 \log c_1 -1-2c_2=0$, which is satisfied only if $c_2=\frac{1}{2}c_1-\frac{1}{2}c_1 \log c_1-\frac{1}{2}$. Consequently, of the two-parameter family of curves (18), we have the following one-parameter family whose members are autopolar:

$$y = c_1 \log (x + 1) - x + \tfrac{1}{2}c_1 - \tfrac{1}{2}c_1 \log c_1 - \tfrac{1}{2}. \tag{19}$$

The particular member of (19) for which $c_1=1$ has the property that the locus of S (the mid-point of P_1P_2) is the line $x+y=0$.

It may be readily verified that, as indicated in §6, the conic of reference, $2\eta=\xi^2$, is the envelope of the one-parameter family of autopolar curves (19).

EXAMPLE 2. $x^3y''+y'^3=0$. The solution is $c_1^2(y-c_2)^2=c_1x^2-1$; and on putting this in (9) we have $2c_2(c_1x^2-1)^{1/2}=0$. Hence $c_2=0$, and of this two-parameter family of hyperbolas there exists the one-parameter autopolar family $c_1x^2-c_1^2y^2=1$. It is readily found from (14) that the locus of Q for these curves is the line $\eta=1/c_1$. And from (15) the locus of R is the line $\eta=-(c_1+1)/c_1$.

EXAMPLE 3. $y''+1=0$. The solution is $y=-\frac{1}{2}x^2+c_1x+c_2$, and this in (9) gives $c_2=-\frac{1}{4}c_1^2$. Hence the parabolas $y=-\frac{1}{2}x^2+c_1x-\frac{1}{4}c_1^2$ are autopolar. It is readily found that for these curves the locus of Q is the line $\xi=\frac{1}{2}c_1$, and that this line is also the locus of S. We also find that the coördinates of the mid-point of the segment QS, $(\frac{1}{2}c_1, \frac{1}{8}c_1^2)$, are identical with the coördinates of P_3, the point of contact of the parabolas with the conic of reference.

EXAMPLE 4. $x+y'=xy'$. The solution of this first-order equation is $y=x+\log (x-1)+c$, and this in (9) gives $c=0$. Hence there is but one member of the family which is autopolar.

EXAMPLE 5. $y'y''+x=0$. From this equation we derive the following one-parameter family of autopolar curves:

$$y = \tfrac{1}{2}c_1^2 \arcsin x/c_1 + \tfrac{1}{2}x(c_1^2 - x^2)^{1/2} - \tfrac{1}{8}c_1^2\pi.$$

There are, of course, many other types of differential equations which are invariant under (2). We mention a few of these: $f(x)+f(y')=F(xy')$, $2y-xy'=f(y')-f(x)$, and $f(y)y''+f(xy'-y)=0$.

The above examples suggest that, in general, we have the following distinction between the autopolar curves obtained from invariant differential equations of the first and second orders. When the solution of such a first-order equation is put in (9), this relation will be satisfied by only a finite number of values of the one arbitrary constant; and hence of this single infinity of curves we shall have but a finite number which are autopolar. When the solution of an invariant second-order equation is put in (9), the relation will be satisfied in many cases by some relation between the two arbitrary constants; that is, from the double infinity of solutions we select a single infinity of autopolar curves. Moreover, in every case, this one-parameter family of autopolar curves envelopes the conic of reference $2\eta=\xi^2$.

There are, however, exceptions to the above. For example, consider the differential equation $(y-x^2)y'+xy=0$, invariant under (2), whose solution is $cx^2+y^2-2cy=0$. On using (9) we find there is no value of c which satisfies this relation. That is, among these conics there are no autopolar curves; but for each conic of the family the polar reciprocal is some other conic of the same family. It may readily be shown that of the above conics, those for which the values of c are negatives of one another are polar reciprocals.

9. **Other dual substitutions.** The dual substitutions for any given conic of reference are easily derived after the manner outlined in Forsyth. And any differential equation which is invariant under these substitutions will have among its solutions certain curves which are autopolar with respect to the given conic.

For example, if the conic of reference be the unit circle, these dual substitutions are[8]

$$y' = -\frac{X}{Y}, \quad x = \frac{-Y'}{Y-XY'}, \quad y = \frac{1}{Y-XY'}, \cdots. \tag{20}$$

Without repeating the argument of §4, the necessary and sufficient condition that a curve $y=F(x)$ be autopolar with respect to the unit circle is

$$\frac{1}{f(x)-xf'(x)} = F\left[\frac{f'(x)}{xf'(x)-f(x)}\right]. \tag{21}$$

[8] Lie und Scheffers, loc. cit., p. 23.

EXAMPLE 6. $xy'+y=0$. This equation is invariant under (20); and its solution, $y=c/x$, when put in (21), satisfies this relation when $c=\pm\frac{1}{2}$. Consequently, the equilateral hyperbolas, $2xy=\pm 1$, are autopolar with respect to the unit circle.

For a complete bibliography on the subject of autopolar curves the reader is asked to consult H. Brocard and T. Lemoyne, *Courbes Géométriques Remarquables*, vol. 1, 1919, pp. 430–432.

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ON THE EQUATION $dy/dx=f(x, y)$[1]

ANDRÉ GLEYZAL

We consider here the differential equation $dy/dx=f(x, y)$, where $f(x, y)$ is a one-valued function defined on an open region R of the xy-plane. By a solution curve of this equation we mean a curve $y=y(x)$ which has a derivative at every point and which satisfies everywhere the differential equation. There are known sufficiency conditions on f for the existence of a one parameter family of solution curves simply covering R. But, as Professor Bamforth mentioned to me orally, there seems to be in the literature no corresponding necessary conditions. We shall prove that one necessary condition is that f be the limit of a sequence of continuous functions.[2]

A curve is the xy-plane will be termed a *continuous function curve* if the points of the curve are the points $(x, \phi(x))$, $a<x<b$, where $\phi(x)$ is a one-valued continuous function defined on an open interval (a, b). An open region R of the xy-plane will be said to be simply covered by a set $\mathcal{J}$ of such curves if:

(1) Every point of R is on one and only one curve of $\mathcal{J}$.

(2) Every curve of $\mathcal{J}$ stretches from boundary to boundary of R; that is, if S is any set of points on a curve C of $\mathcal{J}$, each limit point of S is either itself a point of C or a boundary point of R.

THEOREM. *If an open region R of the xy-plane is simply covered by a set $\mathcal{J}$ of continuous function curves $y=\phi(x)$, then for every point (x_0, y_0) of R there exists an open subregion R_0 of R containing (x_0, y_0) such that the family of curves constituted by the portions of the curves of $\mathcal{J}$ in R_0 is representable by the equation $y=\phi(x, a)$, where ϕ is a continuous function of x and the parameter a.*

PROOF. Let (x_0, y_0) be a point of such a given region R; R_1 a rectangle interior to R with (x_0, y_0) as center; and h a positive number such that the points (x_0, y_0-h) and (x_0, y_0+h) are inside R_1. Since $\mathcal{J}$ simply covers R, there exist curves $y=\phi_1(x)$ and $y=\phi_2(x)$ of $\mathcal{J}$ containing the points (x_0, y_0-h) and (x_0, y_0+h) respectively. Also, the continuity of $\phi_1(x)$ and $\phi_2(x)$ insures the existence of an open interval I containing x_0 such that the points of the curves $y=\phi_1(x)$ and

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[1] I wish to express my thanks to Professor Henry Blumberg for his aid in the preparation of this paper.

[2] Of course, as is well known, the derivative $f'(x)$ of a function $f(x)$ has this property. It may be expected that $f(x, y)$ necessarily has also other properties corresponding to known properties of $f'(x)$.

$y=\phi_2(x)$, with x in I, lie in R_1. If (x_1, x_2) is a closed interval interior to I and containing x_0, we have $\phi_1(x)<\phi_2(x)$ for $x_1\leqq x\leqq x_2$, since $\phi_1(x_0)<\phi_2(x_0)$ and no two curves of $\mathcal{F}$ cross each other. We denote by R_0 the open region of points (x, y) bounded by the parallels $x=x_1$, $x=x_2$, and the curves $y=\phi_1(x)$, $y=\phi_2(x)$. In view of conditions (1) and (2) it readily follows that the interval of definition of any curve $y=\phi(x)$ of $\mathcal{F}$ having points in R_0 contains the entire closed interval (x_1, x_2).

We now define the function $\phi(x, a)$, mentioned in the theorem, by letting $y=\phi(x, a)$ be the equation of the curve of $\mathcal{F}$ passing through the point (x_1, a) on the boundary of R_0. Thus $\phi(x, a)$ is defined in the region $\overline{R}_0$ of the xa-plane, $x_1<x<x_2$, $\phi_1(x_1)<a<\phi_2(x_1)$. The family of curves $y=\phi(x, a)$, where x and the parameter a take on values in $\overline{R}_0$, is identical with the set consisting of the portions of the curves of $\mathcal{F}$ which are interior to R_0. $\phi(x, a)$ is a continuous function of x and a in $\overline{R}_0$. For let (ξ, α) be a point of this region, ϵ any positive number, and α_1, α_2 numbers such that $y=\phi(x, \alpha_1)$, $y=\phi(x, \alpha_2)$ are respectively equations of the curves of $\mathcal{F}$ which pass through the points $(\xi, \phi(\xi, \alpha)-\epsilon')$, $(\xi, \phi(\xi, \alpha)+\epsilon')$ of R_0, ϵ' being a positive number less than $\epsilon/4$. Due to the continuity of $\phi(x, \alpha)$, considered as a function of x, we may find an interval I_1, containing ξ, such that $\phi(x, \alpha_1)$ differs from $\phi(\xi, \alpha_1)$ by an amount less than $\epsilon/4$ for values of x in I_1. Also, we may choose an interval I_2 such that $\phi(x, \alpha_2)$ differs from $\phi(\xi, \alpha_2)$ by less than $\epsilon/4$ for x in I_2. Thus $\phi(x, a)$ differs from $\phi(\xi, \alpha)$ by less than ϵ if $\alpha_1<a<\alpha_2$ and x is in I_1I_2—the interval consisting of the points common to both I_1 and I_2. Consequently, $\phi(x, a)$ is continuous in $\overline{R}_0$, and the theorem is proved.

LEMMA. *The function $\phi(x, a)$ has a continuous inverse function $a(x, y)$ defined on R_0, which satisfies the identity $\phi(x, a(x, y))\equiv y$.*

PROOF. We associate with every point (x, y) of R_0 that value $a=a(x, y)$ such that the curve $y=\phi(x, a)$ passes through the point (x, y). In short, $\phi(x, a(x, y))=y$. The function $a(x, y)$ thus defined is continuous in R_0. For suppose it were discontinuous at a point (ξ, η) of R_0. Since $\phi(x, a)$ is continuous at $(\xi, a(\xi, \eta))$, and properly monotone in a, the composite function $\phi(x, a(x, y))$, considered as a function of x and y, is discontinuous at (ξ, η). But $\phi(x, a(x, y))\equiv y$ for points (x, y) in R_0 and is therefore continuous at the point (ξ, η), contrary to assumption. Therefore $a(x, y)$ is continuous in R_0.

THEOREM. *For the existence of a family $\mathcal{F}$ of continuous function curves simply covering R which are solution curves of the equation*

$dy/dx=f(x, y)$, where R is an open region on which f is defined, it is necessary that f be the limit of a sequence of continuous functions.

PROOF. We consider $f(x, y)$ given, and assume that a family $\mathcal{F}$, as described, exists. If (x_0, y_0) is a point of R there may be determined, as we have shown, an open region R_0 containing the point such that the portions of the curves of $\mathcal{F}$ in R_0 are the curves $y=\phi(x, a)$, $x_1<x<x_2, a_1<a<a_2$. $\phi(x, a)$ has, according to the above lemma, a continuous inverse function $a(x, y)$ defined on R_0 for which $\phi(x, a(x, y))=y$. Let R_0' be the open subregion consisting of the points (x, y) of R_0 where $x_1<x<x_2-k$, k being a positive number less than x_2-x_0. If (ξ, η) is a point of R_0', the equation of the curve of $\mathcal{F}$ which passes through this point is $y=\phi(x, \alpha)$, where $\alpha=a(\xi, \eta)$. By hypothesis, $y=\phi(x, \alpha)$ is a solution curve of the differential equation $dy/dx=f(x, y)$. At (ξ, η) this equation may be written:

$$f(\xi, \eta) = \lim_{n\to\infty} \{\phi(\xi + k/n, a(\xi, \eta) - \phi(\xi, a(\xi, \eta))\}/(k/n)$$

where n is a positive integer. For convenience, we denote the difference quotient present in the right-hand member of this equation by $\psi_n(\xi, \eta)$. Inasmuch as (ξ, η) is a general element of R_0', we have $f(x, y)=\lim_{n\to\infty} \psi_n(x, y)$ in this region. Moreover, $\psi_n(x, y)$ is continuous since $a(x, y)$ is continuous in R_0' and $\phi(x, a)$ is continuous at points $(x, a(x, y))$ and $(x+k/n, a(x, y))$, where (x, y) is in R_0'. We conclude that for every point (x_0, y_0) of R, there exists an open region R_0' containing (x_0, y_0) such that in R_0' the function $f(x, y)$ is the limit of a sequence of continuous functions.

It follows that f is the limit of a sequence of continuous functions in its entire region of definition R. For let P be any perfect subset of the points of the xy-plane containing points of R, and (x_0, y_0) any point of PR. As we have seen, there exists an open region R_0' containing (x_0, y_0) such that, in this region, f is the limit of a sequence of continuous functions. The latter statement implies, it may be shown, that PR_0' has a point of continuity of f with respect to PR_0'.[3] Clearly, this point is a point of continuity of f with respect, not merely to the subset PR_0', but with respect to P, and by Baire's theorem f is the limit of a sequence of continuous functions.

ST MICHAEL'S COLLEGE

[3] For, a necessary and sufficient condition that $f(x, y)$, defined on an open region R of the xy-plane, be the limit of a sequence of continuous functions is that every set PR_0, where P is a perfect set and R_0 is an open subregion of R, have a point of continuity of f with respect to PR_0. This is a slight modification of Baire's theorem and is proved in a paper *On interval functions* which the author is preparing for publication.

NOTE ON INTERPOLATION

G. GRÜNWALD

Let

$$A_n:\ x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)}, \qquad -1 \leqq x_1^{(n)},\ 1 \geqq x_n^{(n)}, \tag{1}$$

denote a set of n distinct points of the interval $-1 \leqq x \leqq +1$. If $f(x)$ is a given function defined in $-1 \leqq x \leqq +1$ we call the polynomial

$$I_n(x; f) = \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x) \tag{2}$$

an interpolation polynomial of $f(x)$ corresponding to the abscissas (1), where for the polynomials $q_1^{(n)}(x),\ q_2^{(n)}(x),\ \cdots,\ q_n^{(n)}(x)$

$$q_k^{(n)}(x_k^{(n)}) = 1, \qquad q_k^{(n)}(x_i^{(n)}) = 0, \qquad i \neq k. \tag{3}$$

Then the polynomial (2) represents a polynomial which assumes the value $f(x_k^{(n)})$ at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$. The polynomials $q_k^{(n)}(x)$ $(k = 1, 2, \cdots, n)$ are called the fundamental polynomials of the interpolation corresponding to the set A_n. We consider the sequence $I_n(x; f)$ $(n = 1, 2, \cdots)$ under the condition of continuity of $f(x)$.

Let $\omega_n(x)$ be a polynomial of degree n, not identically zero, vanishing at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$ and let

$$q_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})} \equiv l_k^{(n)}(x); \tag{4}$$

then (2) represents the nth Lagrange polynomial of $f(x)$ corresponding to the abscissas (1), which is the uniquely determined polynomial of degree $n-1$ which assumes the value $f(x_k^{(n)})$ at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$.

It is known that for a given arbitrary sequence $\{A_n\}$:

$$x_1^{(1)};\ x_1^{(2)},\ x_2^{(2)};\ x_1^{(3)},\ x_2^{(3)},\ x_3^{(3)};\ \cdots;\ x_1^{(n)},\ x_2^{(n)},\ \cdots,\ x_n^{(n)};\ \cdots \tag{5}$$

there exists a continuous function $f(x)$ such that the sequence of the Lagrange interpolation polynomials is not uniformly convergent, even divergent at a preassigned point.[1]

[1] G. Faber, *Über die interpolarische Darstellung stetiger Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 389–408. S. Bernstein, *Sur la limitation des valeurs d'une polynome $P_n(x)$ de degré n sur tout un segment par ses valeurs en $(n+1)$ points du segment*, Bulletin de l'Académie des Sciences de l'URSS, 1931, pp. 1025–1050. In the important special case $x_k^{(n)} = \cos(2k-1)\pi/2n$, that is, for the zeros $\omega_n(x) = \cos n(\arccos x)$ ($\omega_n(x)$ is the nth Tschebycheff polynomial) much more is known. I have proved the existence of a continuous function $f(x)$ for which

Let

$$v_k^{(n)}(x) = 1 - (x - x_k^{(n)}) \frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})}, \tag{6}$$

$$q_k^{(n)}(x) = v_k^{(n)}(x)\{l_k^{(n)}(x)\}^2; \tag{7}$$

then $I_n(x; f)$, the nth Hermite interpolation polynomial of $f(x)$, represents the uniquely determined polynomial of degree $2n-1$ for which $I_n(x_k^{(n)}; f) = f(x_k^{(n)})$ $(k=1, 2, \cdots, n)$ and $I_n'(x_k^{(n)}; f) = 0$ $(k=1, \cdots, n)$. There are sequences $\{A_n\}$ for which the sequence of Hermite interpolation polynomials is uniformly convergent for an arbitrary continuous function.[2]

The question arises whether it is possible to determine, for a given arbitrary sequence $\{A_n\}$, the fundamental polynomials

$$\begin{gathered} q_1^{(1)}(x);\quad q_1^{(2)}(x),\ q_2^{(2)}(x);\quad q_1^{(3)}(x),\ q_2^{(3)}(x),\ q_3^{(3)}(x);\ \cdots; \\ q_1^{(n)}(x),\ q_2^{(n)}(x), \cdots, q_n^{(n)}(x);\ \cdots \end{gathered} \tag{8}$$

so that for an arbitrary continuous function $f(x)$

$$\lim_{n=\infty} I_n(x; f) = \lim_{n=\infty} \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x) = f(x) \tag{9}$$

holds uniformly in $-1 \leqq x \leqq 1$.

For the sequence $\{A_n\}$ we must suppose that the set (5) is everywhere dense in $-1 \leqq x \leqq +1$, that is, $\max_{(i)} (x_{i+1}^{(n)} - x_i^{(n)}) \to 0$ when $n \to \infty$. We prove that this trivial necessary condition is however sufficient to construct the fundamental polynomials (8) so that (9) holds for every continuous $f(x)$.

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the sequence of Lagrange polynomials corresponding to these abscissas is everywhere divergent, even everywhere unbounded. G. Grünwald, *Über Divergenzerscheinungen der Lagrange'schen Interpolationspolynome stetiger Funktionen*, Annals of Mathematics, (2), vol. 37 (1936), pp. 908–918. See also J. Marcinkiewicz, *Sur la divergence des polynomes d'interpolation*, Acta Litterarum ac Scientiarum, Szeged, vol. 8 (1937), pp. 131–135.

[2] Such a sequence is, for example, $x_k^{(n)} = \cos(2k-1)\pi/2n$. See L. Fejér, *Über Weierstrass'sche Approximation, besonders durch Hermite'sche Interpolation*, Mathematische Annalen, vol. 102 (1930), pp. 707–725. It was Fejér who began the investigation of the Hermite interpolation polynomials instead of and beside the Lagrange polynomials, and the fundamental methods and theorems of the theory are due to him. See, for example, L. Fejér, *On the characterisation of some remarkable systems of points of interpolation by means of conjugate points*, American Mathematical Monthly, vol. 41, (1939), pp. 1–14.

Let $\phi_k^{(n)}(x)$ be the continuous function defined in the following way: $\phi_1^{(1)}(x) \equiv 1$; for $n > 1$, and $k \neq 1, n$,

$$\phi_k^{(n)}(x) = \begin{cases} 0 \text{ in } -1 \leqq x \leqq x_{k-1}^{(n)}, \\ \text{linear in } x_{k-1}^{(n)} \leqq x \leqq \tfrac{1}{2}(x_{k-1}^{(n)} + x_k^{(n)}), \\ 1 \text{ in } \tfrac{1}{2}(x_{k-1}^{(n)} + x_k^{(n)}) \leqq x \leqq x_k^{(n)}, \\ \text{linear in } x_k^{(n)} \leqq x \leqq \tfrac{1}{2}(x_k^{(n)} + x_{k+1}^{(n)}), \\ 0 \text{ in } \tfrac{1}{2}(x_k^{(n)} + x_{k+1}^{(n)}) \leqq x \leqq +1; \end{cases}$$

for $k = 1, n$

$$\phi_1^{(n)}(x) = \begin{cases} 1 \text{ in } -1 \leqq x \leqq x_1^{(n)}, \\ \text{linear in } x_1^{(n)} \leqq x \leqq \tfrac{1}{2}(x_1^{(n)} + x_2^{(n)}), \\ 0 \text{ in } \tfrac{1}{2}(x_1^{(n)} + x_2^{(n)}) \leqq x \leqq +1; \end{cases}$$

$$\phi_n^{(n)}(x) = \begin{cases} 0 \text{ in } -1 \leqq x \leqq x_{n-1}^{(n)}, \\ \text{linear in } x_{n-1}^{(n)} \leqq x \leqq \tfrac{1}{2}(x_{n-1}^{(n)} + x_n^{(n)}), \\ 1 \text{ in } \tfrac{1}{2}(x_{n-1}^{(n)} + x_n^{(n)}) \leqq x \leqq +1. \end{cases}$$

We have evidently

$$\sum_{k=1}^{n} \phi_k^{(n)}(x) \equiv 1. \tag{10}$$

Let $q_k^{(n)}(x)$ be a polynomial for which

$$q_k^{(n)}(x_i^{(n)}) = \phi_k^{(n)}(x_i^{(n)}), \qquad i = 1, 2, \cdots, n, \tag{11}$$

$$\left| q_k^{(n)}(x) - \phi_k^{(n)}(x) \right| < 1/n^2, \qquad -1 \leqq x \leqq +1. \tag{12}$$

The existence of such a polynomial follows from a slight modification of the Weierstrass approximation theorem.[3] Equation (3) follows from (11), and from (12) and (10) we have

$$\begin{aligned} \left| \sum_{k=1}^{n} q_k^{(n)}(x) - 1 \right| &= \left| \sum_{k=1}^{n} (q_k^{(n)}(x) - \phi_k^{(n)}(x)) \right| \\ &\leqq \sum_{k=1}^{n} \left| q_k^{(n)}(x) - \phi_k^{(n)}(x) \right| \leqq \frac{1}{n}. \end{aligned} \tag{13}$$

[3] This modification is as follows: Let $f(x)$ be continuous in $-1 \leqq x \leqq +1$ and $x_1, x_2, \cdots, x_n$ given numbers in $-1 \leqq x \leqq +1$. Then for any $\epsilon > 0$ there exists a polynomial $P(x)$ so that $|P(x) - f(x)| < \epsilon$ and $P(x_i) = f(x_i)$, $i = 1, 2, \cdots, n$.

Let x be a fixed number in the interval $-1 \leqq x \leqq +1$ and $\epsilon > 0$. Because of the continuity of the function $f(x)$ if n is large enough and $|x - x_k^{(n)}| < \delta(\epsilon) = \delta$ we have

$$|f(x) - f(x_k^{(n)})| < \epsilon/2;$$

such $x_k^{(n)}$ exist if n is large enough because the sequence is everywhere dense. Then we have from (13)

$$\begin{aligned}
|I_n(x;f) - f(x)| &= \left| \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x) - f(x) \right| \\
&= \left| \sum_{k=1}^{n} (f(x_k^{(n)}) - f(x)) q_k^{(n)}(x) - f(x) + f(x) \sum_{k=1}^{n} q_k^{(n)}(x) \right| \\
&\leqq \sum_{k=1}^{n} |f(x) - f(x_k^{(n)})| \, |q_k^{(n)}(x)| \\
&\quad + |f(x)| \left| \sum_{k=1}^{n} q_k^{(n)}(x) - 1 \right| \qquad (14) \\
&\leqq \sum_{|x - x_k^{(n)}| > \delta} |f(x) - f(x_k^{(n)})| \, |q_k^{(n)}(x)| \\
&\quad + \frac{\epsilon}{2} \sum_{|x - x_k^{(n)}| < \delta} |q_k^{(n)}(x)| + |f(x)|/n \\
&\leqq 2M \sum_{|x - x_k| > \delta} |q_k^{(n)}(x)| + \frac{\epsilon}{2}(1 + 1/n) + M/n,
\end{aligned}$$

where $M = \max_{-1 \leqq x \leqq +1} |f(x)|$. For large n and fixed δ, $\varphi_k^{(n)}(x) = 0$ if $\left| x - x_k^{(n)} \right| > \delta$, so it follows from (12) that $|q_k^{(n)}(x)| < 1/n^2$ if $\left| x - x_k^{(n)} \right| > \delta$, (14) gives for large n

$$|I_n(x;f) - f(x)| < 2M/n + \tfrac{1}{2}\epsilon(1 + 1/n) + M/n < \epsilon,$$

which was to be proved.

BUDAPEST, HUNGARY

ON DERIVATIVES OF ORTHOGONAL POLYNOMIALS. II[1]

H. L. KRALL

It has been shown [1, 2, 7][2] that if $\{y_n(x)\}$ is a set of orthogonal polynomials whose derivatives $\{y_n'(x)\}$ also form an orthogonal set, then the original set $\{y_n(x)\}$ are either Jacobi, Laguerre, or Hermite polynomials. In the papers referred to, the ordinary definition of orthogonal polynomials was used, so of course the result was based on this definition. Since then, however, there has appeared [3, 5][3] a definition of generalized orthogonal polynomials[4] and it is our purpose here to find what G.O.P. sets have G.O.P. derivatives. Besides the classical polynomials, we shall find another class of polynomials which have this property. Of course, this new class will be G.O.P., but not ordinary orthogonal polynomials.

DEFINITION. *Given a set of constants* $\{c_n\}$ *subject to the condition*

$$\Delta_{n+1} = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} \neq 0, \qquad n = 0, 1, \cdots,$$

the polynomials

$$y_n(x) = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

form a set of G.O.P.

It has been shown [5, 4] that there exists a function of bounded variation $\psi(x)$ (hence infinitely many), such that

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[1] Presented to the Society, April 26, 1940, under the title *Orthogonal polynomial solutions of a certain fourth order differential equation.*

[2] The numbers in square brackets refer to the bibliography at the end.

[3] So far as this author knows, the idea of generalized orthogonal polynomials originated with I. M. Sheffer, who suggested its use in [3].

[4] Hereafter we shall use the abbreviation G.O.P. for generalized orthogonal polynomials.

$$c_n = \int_{-\infty}^{+\infty} x^n d\psi(x).$$

If the constants $\{c_n\}$ are real and $\Delta_n > 0$ for all values of n, this definition becomes the usual definition of orthogonal polynomials. In this case one of the $\psi(x)$ will be monotonically non-decreasing.

In this paper we follow the method of Hahn [1] who showed that if $\{y_n(x)\}$ and $\{y_n'(x)\}$ are sets of orthogonal polynomials, the set $\{y_n(x)\}$ satisfies a differential equation of the form[5]

$$(1) \qquad (l_{22}x^2 + l_{21}x + l_{20})y_n''(x) + (l_{11}x + l_{10})y_n'(x) = \lambda_n y_n(x)$$

where the $\{l_{ij}\}$ are constants and $\lambda_n = l_{11}n + l_{22}n(n-1)$. Hahn divided his discussion of (1) according to the roots of the coefficient of $y_n''(x)$. Following his example and making suitable linear transformations, we find that there are four distinct cases of (1) to be considered:

$$(2) \qquad x(1-x)y_n''(x) + \{\alpha - (\alpha+\beta)x\}y_n'(x) + n(\alpha+\beta+n-1)y_n(x) = 0,$$

$$(3) \qquad xy_n''(x) + (\alpha - hx)y_n'(x) + hny_n(x) = 0,$$

$$(4) \qquad y_n''(x) - 2xy_n'(x) + 2ny_n(x) = 0,$$

$$(5) \qquad x^2y_n''(x) + (kx + l)y_n'(x) - n(k+n-1)y_n(x) = 0.$$

By using the ordinary definition, it readily follows that the only orthogonal polynomial solutions of these equations are the Jacobi polynomials (with $\alpha, \beta > 0$) from (2), the Laguerre polynomials (with $\alpha > 0$) from (3), and the Hermite polynomials from (4).

We now examine these differential equations for G.O.P. solutions. For this discussion, we need a result from [3]:

In order that there exists a set of G.O.P. as solutions of (1), it is necessary and sufficient that the "moments" $\{c_n\}$ satisfy the conditions

$$(6) \qquad \begin{aligned} &\{l_{11} + l_{22}(n-1)\}c_n + \{l_{10} + l_{21}(n-1)\}c_{n-1} + l_{20}(n-1)c_{n-2} = 0, && n = 1, 2, \cdots, \\ &\Delta_{n+1} \neq 0, && n = 0, 1, 2, \cdots. \end{aligned}$$

Discussion of (2). Here, the recurrence relation (6) becomes $(n+\alpha+\beta-1)c_n - (n+\alpha-1)c_{n-1} = 0$, $n = 1, 2, \cdots$. From this we get

[5] This part of Hahn's work applies to G.O.P. His notation has been changed to conform to [3].

$$c_n = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)c_0}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)},$$

$$\Delta_{n+1} = \prod_{i=0}^{n} \frac{\Gamma(i+1)\Gamma(\alpha+i)\Gamma(\beta+i)\Gamma(\alpha+\beta)c_0}{\Gamma(\alpha+\beta+i+n)\Gamma(\alpha)\Gamma(\beta)}.$$

Accordingly, if $\alpha, \beta, \alpha+\beta \neq 0, -1, -2, \cdots$, the differential equation (3) has a set of G.O.P. as solutions. However, it does not seem to be easy to write out the weight function when either α or β is less than zero. These polynomials have been discussed by Szegö [6].

Discussion of (3). In this case we have

$$hc_n - (n+\alpha-1)c_{n-1} = 0,$$

$$c_n = \frac{\Gamma(\alpha+n)c_0}{h^n\Gamma(\alpha)}, \qquad \Delta_{n+1} = \prod_{i=0}^{n} \frac{\Gamma(i+1)\Gamma(\alpha+i)c_0}{h^{2i}\Gamma(\alpha)}.$$

Accordingly, if $h \neq 0$, and $\alpha \neq 0, -1, -2, \cdots$, the differential equation (3) has a set of G.O.P. as solutions. These polynomials have also been discussed by Szegö [6].

Discussion of (4). There are no exceptional cases here. The solutions are Hermite polynomials.

Discussion of (5). In this case we have

$$(k+n-1)c_n + lc_{n-1} = 0,$$

$$c_n = \frac{l^n\Gamma(k)c_0}{\Gamma(k+n)}, \qquad \Delta_{n+1} = \prod_{i=0}^{n} \frac{(-l^2)^i\Gamma(k)\Gamma(i+1)c_0}{\Gamma(k+i+n)}.$$

Accordingly, if $l \neq 0$, and $k \neq 0, -1, -2, \cdots$, the differential equation (5) has a set of G.O.P. as solutions. For no value of k will Δ_{n+1} be greater than zero for all values of n. These polynomials were discussed by Hahn [1] to the extent of showing that they are not orthogonal polynomials in the ordinary sense. They are not mentioned by any of the other authors.

Returning to (1) and differentiating, we get

$$(l_{22}x^2 + l_{21}x + l_{20})y_n'''(x) + \{(l_{11}+2l_{22})x + l_{10} + l_{21}\}y_n''(x) = (\lambda_n - l_{11})y_n'(x). \tag{7}$$

Thus, if a set of polynomials $\{y_n(x)\}$ satisfies (1), the set of derivatives $\{y_n'(x)\}$ satisfies the equation

$$(l_{22}x^2 + l_{21}x + l_{20})\phi''(x) + \{(l_{11}+2l_{22})x + l_{10} + l_{21}\}\phi'(x) = (\lambda_n - l_{11})\phi(x). \tag{8}$$

Since (7) is the same type as (1) and since (7) will not be an exceptional case if (1) is not, it follows that the only sets of G.O.P. whose derivatives are again G.O.P. are:

(i) the Jacobi polynomials with α, β, $\alpha+\beta \neq 0, -1, -2, \cdots$,

(ii) the Laguerre polynomials with $h \neq 0$ and $\alpha \neq 0, -1, -2, \cdots$,

(iii) the Hermite polynomials,

(iv) the polynomial solutions of (5) with $l \neq 0$, and $k \neq 0, -1, -2, \cdots$.

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NOTE ON PROBABILITY IMPLICATION

HANS REICHENBACH

In a recently published paper[1] J. C. C. McKinsey has pointed out some difficulties which arise from Axiom I of my theory of probability implication.[2] This axiom states the unambiguity of the degree p of a given probability implication $(O \ni_p P)$ for the case that the class O is not empty, a condition formulated by $(\overline{\overline{O}})$, but postulates ambiguity of p in case of an empty class O, this condition being formulated by $(\overline{O})$. The latter ambiguity is necessary for probability implication because of the relation to Russell's material implication.[3] From the proof published by McKinsey we can infer that this ambiguity has to be restricted to values of p between 0 and 1, limits included, in correspondence with the same restriction holding for the unambiguous degree p of probability in cases of a non-empty class O, formulated by me in (8, §13).[4] That this general restriction is derivable from Axiom II, 2 is obvious as this axiom contains O and p as free variables and therefore states the restriction for all classes O and all values p.

A further objection, which was already indicated in a footnote of McKinsey's paper, has been presented to me in a letter by the referee of this journal, Mr. S. C. Kleene. This objection shows that if the ambiguity of degrees of probability for empty classes O is assumed, it can be proved that this ambiguity cannot be restricted to the limits 0 to 1.

This proof is connected with the theorem of addition (Axiom III) which reads[5]

III. $(O \ni_p P) . (O \ni_q Q) . (O . P \supset \overline{Q}) \supset (\exists r)(O \ni_r P \vee Q) . (r = p + q)$.

The condition $r \leqq 1$ implies that $p+q \leqq 1$. If we demand $r \leqq 1$ only for non-empty classes O, the mentioned restriction for p and q, which

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[1] This Bulletin, vol. 45 (1939), pp. 799–800.

[2] Published in *Wahrscheinlichkeitslehre*, Leiden, 1935, §§12–14. My further quotations refer to this book.

[3] Page 66.

[4] To avoid misunderstandings let me add here the remark that this relation is meant only for the case that the probability $W(O, P)$ *exists*, and would be written in the implicational writing

$$[(\exists x)\ (O \ni_x P)] \supset [(\exists y)\ (O \ni_y P) . (0 \leqq y \leqq 1)].$$

If O is not empty and therefore the probability has only one value, this means that this value is restricted to the limits 0 to 1, limits included.

[5] I write here the existential operator on the right-hand side because the abbreviation introduced on page 62, according to which the existential operator is omitted in the corresponding formula of my book, may be misleading.

is stronger than (8, §13), need not be stated in the implicans of the axiom because we then rather infer from III that the two conditions

$$(1) \qquad p + q > 1, \qquad (O.P \supset \overline{Q})$$

are incompatible. Following the principle that numerical values leading to contradictions are to be excluded, we can thus deduce all numerical restrictions for the values of probabilities; in (15, §19) I have developed a formula which states these restrictions for all possible cases. The incompatibility of the two formulae (1) for non-empty classes O is included in (15, §19) because this relation shows that if $p+q>1$ we have $W(O.P, Q)>0$, which in consideration of (9, §13) contradicts $(O.P \supset \overline{Q})$ in case O is not empty. However in applying the explained principle we have to be sure that the excluded numerical values cannot be introduced by means of other ways of deduction. It turns out that this is the case for an empty class O. In this case, that is for $(\overline{O})$, we can for instance assume in III the numerical values $p=1$ and $q=1$ whereas the condition $(O.P \supset \overline{Q})$ is also satisfied; the latter follows from the properties of material implication according to which a false proposition implies every proposition. It follows that in this case r can be greater than 1.

These difficulties are eliminated if we introduce in Axiom II,2 the condition that O is not empty, and write this axiom:

II,2. $\overline{(\overline{O})}.(O \ni_p P) \supset (p \geqq 0)$.

With the qualification $\overline{(\overline{O})}$ in the implicans we renounce the unnecessary extension of numerical restrictions to probability values referring to empty classes. It is without danger to assume that in case O is empty the probability p of $(O \ni_p P)$ can be greater than 1 or smaller than 0, and we shall make use of this liberty if it helps us to escape contradictions. The range of the variables expressed by small letters "p," "q," and so on, extends therefore through all real numbers, the necessary restrictions for non-empty classes O being expressed by the Axiom II,2.

As far as I see we need not introduce any further qualifications within the system of probability implication. I should like however to add some remarks concerning the use of the functor $W(\ )$. A functor is usually conceived as being unambiguous; thus $W(O, P)$ would mean "*the* probability from O to P." Our symbol has this meaning only in case O is not empty. To include the case of O being empty we have to translate every formula of the functor-writing into an existence statement such as explained on pages 61–62, or in the examples on pages 69–70 and page 73. The formula then means, in accordance

with the rule of existence: if $n-1$ of the n probabilities of the formula exist and have among their values the values $p, q, \cdots$, then also the nth probability exists and has among its values a value w such that the $p, q, \cdots, w$ satisfy the equation expressed in the formula. It is obvious that for non-empty classes O this interpretation is identical with the meaning of unambiguous functors. For the case of empty classes O however it also will lead to consistent interpretations although of course in this case the formulae actually state nothing, as all numerical values of the probabilities then are possible. It follows that we need not add the condition $\overline{(O)}$ to any of the formulae of the functor-writing.

For all practical purposes it is convenient to interpret the functor formulae as referring to a non-empty class O, and then to interpret the functor in the usual way as meaning "*the* probability from . . . to . . . ." Here the term O is defined as that term which occurs in the first place of every probability expression of a formula. It is interesting that this interpretation can also be carried through if some of the other classes are empty, even if they appear in the first place of the probability functor; it turns out that each formula can be written in a form in which the indeterminate probability values are multiplied by 0 such as in (3), p. 73, if we assume P to be empty.

I am much indebted to Mr. J. C. C. McKinsey and to Mr. S. C. Kleene for having pointed out the necessity of the correction of my axioms of probability implication with respect to empty classes O.

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ON THE ANALOGUE FOR DIFFERENTIAL EQUATIONS OF THE HILBERT-NETTO THEOREM

RICHARD COHN

If

$$F_1, \cdots, F_r \tag{1}$$

is a finite system of differential polynomials in the unknowns $y_1, \cdots, y_n$, and if G is a differential polynomial which is annulled by every solution of the system (1), some power G^p of G is a linear combination of the F_i and their derivatives of various orders, with differential polynomials for coefficients. This analogue of the Hilbert-Netto theorem was proved by J. F. Ritt[1] for forms with meromorphic coefficients, and by H. W. Raudenbush[2] for the case of coefficients belonging to an abstract differential field. In these proofs it is shown that the denial of the existence of the exponent p, above, of G leads to a contradiction; no constructive method for obtaining admissible values of p is given. The object of the present note is to present a new proof of the analogue, for the case of meromorphic coefficients, which is entirely constructive and produces a definite G^p as described above.

Our proof will be based on the considerations in Chapters V and VII of A.D.E. In Chapter VII, the problem of obtaining G is reduced to the problem of determining unity as a linear combination of the F_i in (1) and their derivatives, in the case in which (1) has no solutions. In Chapter V it is shown how to decide, in a finite number of steps, whether or not (1) has solutions. Our problem thus assumes the following form: *Given that* (1) *has no solutions, it is required to express unity as a linear combination of the* F_i *and their derivatives.*

We assume that (1) has no solutions and proceed to examine the algorithm developed in §§65–67 of A.D.E. Adjoining to (1) a finite number of linear combinations of the F_i and their derivatives, we obtain a system Σ, devoid of solutions, with a basic set

$$A_1, \cdots, A_q \tag{2}$$

which has the property that the remainder of every form in Σ with respect to (2) is zero. If (2) consists of a single form A which is an

[1] Ritt, J. F., *Differential Equations from the Algebraic Standpoint*, chap. 7, referred to below as A.D.E. American Mathematical Society Colloquium Publications, vol. 14, 1932.

[2] Raudenbush, H. W., *Ideal theory and algebraic differential equations*, Transactions of this Society, vol. 36 (1934), pp. 361–368.

element of $\mathfrak{F}$, the coefficient field of our forms, we secure immediately a representation of the type desired for unity. Let us suppose that this is not the case. Then (2), considered as a set of simple forms, cannot be a basic set of a prime system; if it were, (1) would possess solutions (A.D.E., §65). Thus there must exist, for some $j \leqq q$, an identity

$$(3) \qquad J_1^{\mu_1} \cdots J_{j-1}^{\mu_{j-1}}(SA_j - H_1H_2) - L_1A_1 - \cdots - L_{j-1}A_{j-1} = 0,$$

where J_i is the initial of A_i, $i=1, \cdots, j-1$; and where H_1 and H_2 are reduced with respect to $A_1, \cdots, A_j$. Let $\Lambda_1^{(k)}$, $k=1, \cdots, j+1$, represent the systems $\Sigma+J_1, \cdots, \Sigma+J_{j-1}, \Sigma+H_1, \Sigma+H_2$, respectively. We treat each $\Lambda_1^{(k)}$ as (1) was treated. The adjunction of a finite number of forms to any $\Lambda_1^{(k)}$ produces a system $\Sigma_1^{(k)}$, with no solutions, and with basic sets lower than (2) which furnish zero remainders for the forms in $\Sigma_1^{(k)}$.

Let us suppose that each $\Sigma_1^{(k)}$ contains an element of $\mathfrak{F}$ different from zero. We see on examining these systems that there exist relations, procurable by constructive methods,

$$(4) \qquad 1 = P + M_0H_1 + M_1H_1' + \cdots,$$

$$(5) \qquad 1 = Q + N_0H_2 + N_1H_2' + \cdots,$$

$$(6) \qquad 1 = R_i + S_{i0}J_i + S_{i1}J_i' + \cdots, \qquad i = 1, \cdots, j-1,$$

accents indicating differentiation, where P, Q, and the R_i are linear in the F_i and their derivatives.

We equate to unity the product of the right-hand members of (4), (5), and the equations (6). If both sides of the resulting equation are raised to a sufficiently high power, determinable in advance, we secure, as Raudenbush has shown, a relation of the type

$$1 = L + T_0V + T_1V' + \cdots,$$

accents indicating differentiation, where L is linear in the F_i and their derivatives, and $V = J_1^{\mu_1} \cdots J_{j-1}^{\mu_{j-1}}H_1H_2$.[3] From (3) we see that V can be obtained as a linear expression in the F_i and their derivatives. We thus have such an expression as we are seeking for unity.

If, on the other hand, some system $\Sigma_1^{(j)}$ does not contain a nonzero element in $\mathfrak{F}$, we apply to it the entire process applied to Σ. We form in this way systems with basic sets lower than those of $\Sigma_1^{(j)}$. The systems thus formed for the various $\Sigma_1^{(k)}$ receiving our present treatment will be called, with no attempt to describe their complete history, systems Σ_2. In each Σ_2 a basic set yields only zero remainders.

[3] Ritt, J. F., *Algebraic aspects of the theory of differential equations*, Semicentennial Publications of the American Mathematical Society, vol. 2, p. 44.

Let us suppose that each Σ_2 contains a nonzero element in $\mathcal{J}$. What precedes shows that, for each $\Sigma_1^{(\rho)}$ as above, unity is linear in the forms of $\Sigma_1^{(\rho)}$ and their derivatives; this, again, gives the expression which we are seeking for unity.

If there are Σ_2 which contain no nonzero element in $\mathcal{J}$, we give them the treatment which is now familiar. By §67 of A.D.E., we know that our process can continue for only a finite number of steps, so that the possibility of determining for unity an expression of the type desired is established.

COLUMBIA UNIVERSITY

ON A CONVERGENCE THEOREM FOR THE LAGRANGE INTERPOLATION POLYNOMIALS

G. GRÜNWALD

The unique polynomial of degree $(n-1)$ assuming the values $f(x_1), \cdots, f(x_n)$ at the abscissas $x_1, x_2, \cdots, x_n$, respectively, is given by the Lagrange interpolation formula

$$L_n(f) = f(x_1)l_1(x) + f(x_2)l_2(x) + \cdots + f(x_n)l_n(x). \tag{1}$$

Here

$$l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \qquad k = 1, 2, \cdots, n \tag{2}$$

(fundamental polynomials of the Lagrange interpolation), and the polynomial $\omega(x)$ is defined by

$$\omega(x) = c(x - x_1)(x - x_2) \cdots (x - x_n), \tag{3}$$

where c denotes an arbitrary constant not equal to zero. It is known and easy to verify that

$$l_1(x) + l_2(x) + \cdots + l_n(x) \equiv 1. \tag{4}$$

In the Lagrange interpolation formula let

$$x_k = x_k^{(n)} = \cos(2k - 1)\pi/2n = \cos\theta_k^{(n)} \tag{5}$$

which implies

$$\omega(x) = T_n(x) = \cos(n \arccos x) = \cos n\theta, \qquad \cos\theta = x \tag{6}$$

(Tchebyscheff polynomial). In this case we have

$$l_k(x) = l_k[\theta] = (-1)^{k+1}\frac{\cos n\theta \sin\theta_k^{(n)}}{n(\cos\theta - \cos\theta_k^{(n)})}, \tag{7}$$

$$k = 1, 2, \cdots, n;\ x = \cos\theta;$$

and

$$L_n(f) = L_n[f;\theta] = \sum_{k=1}^{n} f(\cos\theta_k^{(n)})(-1)^{k+1}\frac{\cos n\theta \sin\theta_k^{(n)}}{n(\cos\theta - \cos\theta_k^{(n)})}, \tag{8}$$

$$x = \cos\theta.$$

Suppose $f(x)$ to be a continuous function; then it is known that

the sequence $L_n(f)$, $n=1, 2, \cdots$, is not convergent[1] for all $f(x)$. We may even find a continuous function $f_1(x)$ such that the sequence $L_n(f_1)$, $n=1, 2, \cdots$, is divergent for all points of the interval $-1 \leqq x \leqq +1$.[2]

Therefore it is interesting to prove the following theorem:

THEOREM. *Let $f(x)$ be a continuous function in the interval $-1 \leqq x \leqq +1$; then*

$$\lim_{n\to\infty} \tfrac{1}{2}\{L_n[f; \theta - \pi/2n] + L_n[f; \theta + \pi/2n]\} = f(x), \qquad x = \cos\theta, \tag{9}$$

and the convergence is uniform in the whole interval $-1 \leqq x \leqq +1$.

Between the interpolation polynomials (8) and the partial sums $s_{n-1}(f)$ of the Fourier series of the even function $f(x)$ there is a far reaching analogy. We mention here only the following. On one hand, it is easy to verify that

$$L_n(f) = c_0 + c_1 \cos\theta + \cdots + c_{n-1}\cos(n-1)\theta, \qquad x = \cos\theta, \tag{10}$$

where

$$c_0 = \frac{1}{n}\sum_{k=1}^{n} f(\cos\theta_k^{(n)}), \qquad c_r = \frac{2}{n}\sum_{k=1}^{n} f(\cos\theta_k^{(n)})\cos r\theta_k^{(n)}, \tag{11}$$
$$r = 1, 2, \cdots, n-1.$$

On the other hand,

$$s_{n-1}(f) = a_0 + a_1\cos\theta + \cdots + a_{n-1}\cos(n-1)\theta, \tag{12}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\cos\theta)d\theta, \qquad a_r = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos r\theta\, d\theta, \tag{13}$$
$$r = 1, 2, \cdots, n-1.$$

Our theorem is analogous with the well known theorem of Rogosinski in the theory of Fourier series.

We first prove the following lemma.

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[1] G. Faber, *Über die interpolatorische Darstellung stetiger Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 190–210. S. Bernstein, *Sur la limitation des valeurs* $\cdots$, Bulletin de l'Académie des Sciences de l'URSS, 1931, pp. 1025–1050.

[2] G. Grünwald, *Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen*, Annals of Mathematics, (2), vol. 37 (1936), pp. 908–918. See also J. Marcinkiewicz, *Sur la divergence des polynomes d'interpolation*, Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Iosephinae, vol. 8 (1937), pp. 131–135.

LEMMA.

$$\frac{1}{2}\sum_{k=1}^{n}\left|\, l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\,\right|<c_1,$$

where $c_1>0$ is an absolute constant.

From (7) we have, for $\theta\neq\theta_k^{(n)}\pm\pi/2n$,

$$\begin{aligned}
&\tfrac{1}{2}(l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n])\\
&\quad=\tfrac{1}{2}(-1)^{k+1}\left(\frac{\cos n(\theta-\pi/2n)\sin\theta_k^{(n)}}{n(\cos(\theta-\pi/2n)-\cos\theta_k^{(n)})}\right.\\
&\qquad\left.+\frac{\cos n(\theta+\pi/2n)\sin\theta_k^{(n)}}{n(\cos(\theta+\pi/2n)-\cos\theta_k^{(n)})}\right) \qquad (14)\\
&\quad=\tfrac{1}{2}(-1)^{k+1}\frac{\sin\theta_k^{(n)}\sin n\theta\sin\theta\sin\pi/2n}{4n\sin\tfrac{1}{2}(\theta+\theta_k^{(n)}+\pi/2n)\sin\tfrac{1}{2}(\theta-\theta_k^{(n)}+\pi/2n)}\\
&\qquad\cdot\frac{1}{\sin\tfrac{1}{2}(\theta+\theta_k^{(n)}-\pi/2n)\sin\tfrac{1}{2}(\theta-\theta_k^{(n)}-\pi/2n)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\,\tfrac{1}{2}(l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n])\,\right|\\
&\quad\leqq\frac{1}{2n}\left|\frac{\sin\pi/2n}{\sin\tfrac{1}{2}(\theta-\theta_k^{(n)}+\pi/2n)\sin\tfrac{1}{2}(\theta-\theta_k^{(n)}-\pi/2n)}\right|\\
&\quad\leqq\frac{1}{2n}\frac{\pi/2n}{(2/\pi)\left|\,\tfrac{1}{2}(\theta-\theta_k^{(n)}+\pi/2n)\,\right|\,(2/\pi)\left|\,\tfrac{1}{2}(\theta-\theta_k^{(n)}-\pi/2n)\,\right|} \qquad (15)\\
&\quad\leqq\frac{\pi^3}{4n^2}\frac{1}{(\theta-\theta_k^{(n)}-\pi/2n)^2}.
\end{aligned}$$

Now let θ be fixed and

$$(16)\quad 1>x_1>x_2>\cdots>x_j>x=\cos\theta>x_{j+1}>\cdots>x_n>-1.$$

It is known that[3]

$$\left|\,l_k(x)\,\right|<4/\pi,\quad k=1,2,\cdots,n;\ n=1,2,\cdots;\ -1\leqq x\leqq+1.$$

Thus

[3] P. Erdös and G. Grünwald, *Note on an elementary problem of interpolation*, this Bulletin, vol. 44 (1938), pp. 515–578. This bound is the best possible; Fejér proved earlier $|l_n(x)|<2^{1/2}$. See L. Fejér, *Lagrangesche Interpolation und die zugehörigen konjugierten Punkte*, Mathematische Annalen, vol. 106 (1932), pp. 1–55.

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{n}\left|\, l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\,\right| \\
&\leqq \frac{16}{\pi}+\frac{1}{2}\sum_{1\leqq k\leqq n,\,k\neq j-2,j-1,j,j+1}\left|\, l_k[\theta-\pi/2n]+l_k[\theta+\pi 2n]\,\right| \\
&\leqq \frac{16}{\pi}+\sum_{1\leqq k\leqq n,\,k\neq j-2,j-1,j,j+1}\frac{\pi^3}{4n_2}\,\frac{1}{(\theta-\theta_k^{(n)}-\pi/2n)^2} \\
&\leqq \frac{16}{\pi}+\frac{\pi^3}{4n^2}\sum_{k=1}^{\infty}\left(\frac{2n}{\pi}\right)^2\frac{1}{k^2}=c_1.
\end{aligned}
\tag{17}
$$

If $\theta=0,\ \pi$, the same inequality evidently holds.

From (17) we obtain for sufficiently large n, $\delta>0$ fixed,

$$
\sum_{1\leqq k\leqq n,\,|\theta-\theta_k^{(n)}|>\delta}\tfrac{1}{2}\left|\, l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\,\right|=O(1/n). \tag{18}
$$

Now we are in the position to prove our theorem. Let $\epsilon>0$ be a fixed number. The identity (4) gives

$$
\frac{1}{2}\sum_{k=1}^{n}\left(l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\right)=1; \tag{19}
$$

hence for a fixed $x=\cos\theta$

$$
\begin{aligned}
&\tfrac{1}{2}\left\{L_n[f;\theta-\pi/2n]+L_n[f;\theta+\pi/2n]\right\}-f(\cos\theta) \\
&\quad=\sum_{k=1}^{n}\tfrac{1}{2}(f(\cos\theta_k^{(n)})-f(\cos\theta))(l_k[\theta+\pi/2n]+l_k[\theta+\pi/2n]).
\end{aligned}
\tag{20}
$$

The function $f(\cos\theta)$ is continuous; thus we can find a positive number δ such that

$$
\left|\, f(\cos\theta)-f(\cos\theta_k^{(n)})\,\right|<\epsilon/2c_1 \tag{21}
$$

whenever $\left|\theta-\theta_k^{(n)}\right|<\delta$. From (20) and (21) we have

$$
\begin{aligned}
\Delta&=\left|\,\tfrac{1}{2}\left\{L_n[f;\theta-\pi/2n]+L_n[f;\theta+\pi/2n]\right\}-f(\cos\theta)\,\right| \\
&=\Bigg|\sum_{1\leqq k\leqq n,\,|\theta-\theta_k^{(n)}|<\delta}\tfrac{1}{2}(f(\cos\theta_k^{(n)})-f(\cos\theta))(l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]) \\
&\quad+\sum_{1\leqq k\leqq n,\,|\theta-\theta_k^{(n)}|>\delta}\tfrac{1}{2}(f(\cos\theta_k^{(n)})-f(\cos\theta))(l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n])\Bigg| \\
&\leqq\frac{\epsilon}{2c_1}\sum_{1\leqq k\leqq n,\,|\theta-\theta_k^{(n)}|<\delta}\tfrac{1}{2}\left|\, l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\,\right| \\
&\quad+\sum_{1\leqq k\leqq n,\,|\theta-\theta_k^{(n)}|>\delta}\tfrac{1}{2}\left|\, f(\cos\theta_k^{(n)})-f(\cos\theta)\,\right|\left|\, l_k[\theta-\pi/2n]+l_k[\theta+\pi/2n]\,\right|
\end{aligned}
\tag{22}
$$

and by the lemma and (18) for sufficiently large n

$$\Delta \leqq \frac{\epsilon}{2c_1} c_1 + \sum_{1 \leqq k \leqq n, |\theta - \theta_k^{(n)}| > \delta} \tfrac{1}{2} 2M \left| l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n] \right|$$
$$< \epsilon/2 + MO(1/n) < \epsilon,$$

where $M = \max_{-1 \leqq x \leqq +1} |f(x)|$, and this proves our theorem.

BUDAPEST, HUNGARY

DISCONTINUOUS CONVEX SOLUTIONS OF DIFFERENCE EQUATIONS[1]

FRITZ JOHN

This paper contains some conditions for continuity of convex solutions of a difference equation.

A function $f(x)$ defined for $a \leqq x \leqq b$ is *convex*, if

$$\left(\frac{x+y}{2}\right) \leqq \frac{f(x)+f(y)}{2}. \tag{1}$$

If $f(x)$ is convex and bounded from above in $a \leqq x \leqq b$, then $f(x)$ is continuous (see Bernstein [1, p. 422]).[2] If $f(x)$ is convex in $a \leqq x \leqq b$ and y a fixed number with $a < y < b$, let the function $\phi_y(x)$ be defined by

$$\phi_y(x) = \lim_{\alpha \to x-y} f(y + \alpha),$$

where α assumes *rational* values only. Then $\phi_y(x)$ is uniquely defined, continuous, and convex for $a < x < b$ (F. Bernstein [1, p. 431, Theorem 7]); moreover $\phi_y(x) = f(x)$ for rational $y - x$.

THEOREM 1. *If there exists at most one continuous convex solution of the difference equation*

$$F(x, f(x), f(x+1), \cdots, f(x+n)) = g(x), \qquad x > 0, \tag{2}$$

where F and g are continuous functions of their arguments, then there exist no discontinuous convex solutions.

PROOF. If $f(x)$ is a convex solution, then, for $x - y$ rational,

$$F(x, \phi_y(x), \phi_y(x+1), \cdots, \phi_y(x+n)) = g(x);$$

---

[1] Presented to the Society, September 12, 1940.

[2] The numbers in brackets refer to the bibliography.

as both members of this equation are continuous functions of x, it holds for *all* $x>0$. As there is at most one continuous convex solution, we have

$$\phi_y(x) = \phi_z(x)$$

for all positive z, y, x. As $\phi_z(z)=f(z)$, we see that $f(z)$ is identical with the continuous function $\phi_y(z)$ for all positive z.

THEOREM 2. *If the difference equation* (2) *has at most one solution, which is monotone for sufficiently large* x, *then* (2) *has at most one convex solution, and that solution will be continuous.*

PROOF. Every continuous convex solution is monotone for sufficiently large x. Apply Theorem 1.

THEOREM 3. *A difference equation of the form*

$$\prod_{k=0}^{n} (f(x+k))^{a_k} = g(x), \qquad x>0, \tag{3}$$

(a_k *real constants*) *has at most one convex solution, if*

(1) *all roots of the equation*

$$\sum_{k=0}^{n} a_k x^k = 0$$

are simple and of absolute value 1,

$$a_n > 0, \qquad \sum_{k=0}^{n} a_k \neq 0,$$

(2) $g(x)\neq 0$ *and continuous,*

(3) $\lim_{x\to\infty} (\log |g(x)|)/x = 0$,

(4) $\lim_{x\to\infty} (\log |g(x)|)/\log x \neq \sum_k a_k$.

PROOF. Assumption (3) implies

$$\prod_{k=0}^{n} |f(x+k)|^{a_k} = |g(x)|.$$

From assumption (1) above and the lemma proved below, it follows that there are *non-negative* constants b_l and c_r, such that

$$\prod_l |g(x+l)|^{b_l} = \prod_l \left[\prod_k |f(x+k+l)|^{a_k} \right]^{b_l} = \prod_r |f(x+r)|^{c_r}.$$

For every continuous convex solution $f(x)$ of (3), $|f(x)|$ is either monotone non-decreasing or monotone decreasing for sufficiently large x. Which of these alternatives takes place is determined by $g(x)$, as in the first case

$$\prod_l |g(x+l)|^{b_l}$$

is monotone non-decreasing, and in the second case monotone decreasing for sufficiently large x. If $|f(x)|$ is monotone non-decreasing, we have in $F(x)=\log|f(x)|$ a non-decreasing solution of the equation

$$\sum_k a_k F(x+k) = \log|g(x)|.$$

It follows from our assumptions that such a solution $F(x)$ is uniquely determined for all sufficiently large x, and hence for all x (John [3, p. 183]). If $|f(x)|$ is monotone decreasing for sufficiently large x, then $F(x)=-\log|f(x)|$ is an increasing solution of

$$\sum_k a_k F(x+k) = -\log|g(x)|,$$

and hence uniquely determined. Thus for any continuous convex solution $f(x)$ of (3), $|f(x)|$ is uniquely determined. Then $f(x)$ is uniquely determined as well, unless $f(x)$ is linear for sufficiently large x; but if $f(x)$ is linear for large x,

$$\lim_{x\to\infty} \frac{\log|g(x)|}{\log x} = \lim_{x\to\infty} \sum_k a_k \frac{\log|f(x+k)|}{\log x} = \sum_k a_k$$

contrary to assumption. Thus there exists at most one continuous convex solution, and hence at most one convex solution.

Example. The equation

$$f(x+1)\cdot f(x) = x^{-p}, \qquad x>0,\ p>0,$$

satisfies the assumptions of Theorem 3 and hence has at most one convex solution (proved by A. E. Meyer [4] for $p=1$, for general p by H. P. Thielman [5]). The convex solution is found to be

$$B(\tfrac{1}{2}x, \tfrac{1}{2})^p/(2\pi)^{p/2}.$$

LEMMA. *If $\phi(x)=\sum_{k=0}^{n} a_k x^k$ is a polynomial, such that*

(1) $a_n>0$,

(2) $\phi(x)$ *has no positive real roots,*

then there are polynomials $\psi(x)$ and $\sigma(x)$ with non-negative coefficients, such that

$$\phi(x)\cdot\psi(x) = \sigma(x).$$

PROOF. $\phi(x)$ can be factored in the form

$$\phi(x) = a_n \prod_l (x + \alpha_l) \prod_k (x^2 + 2\beta_k x + \gamma_k)$$

where $\alpha_l \geqq 0$, $\gamma_k > \beta_k^2$. Hence it is sufficient to prove the lemma for the case that

$$\phi(x) = x^2 + 2\beta x + \gamma$$

and $\beta^2 < \gamma$. We define θ with $0 < \theta < \pi$ by

$$\cos\theta = -\beta/\gamma^{1/2}.$$

Let the non-negative *integer* s be determined by

$$\frac{\pi}{s+2} \leqq \theta < \frac{\pi}{s+1}.$$

Put

$$\psi(x) = \frac{(\gamma^{1/2})^{s+2} \sin(s+1)\theta - x(\gamma^{1/2})^{s+1} \sin(s+2)\theta + x^{s+2} \sin\theta}{[x^2 + 2\beta x + \gamma] \cdot \sin\theta}$$

$$= \sum_{k=0}^{s} (\gamma^{1/2})^k \frac{\sin(k+1)\theta}{\sin\theta} x^{s-k}.$$

$\psi(x)$ and $(x^2+2\beta x+\gamma)\cdot\psi(x)$ obviously are polynomials with non-negative coefficients.

THEOREM 4. *The difference equation*

$$\sum_{k=0}^{n} a_k f(x+k) = g(x), \qquad x > 0, \tag{4}$$

has no discontinuous convex solutions, if

(a) *$g(x)$ is bounded from above in every positive interval,*

(b) *$a_n > 0$,*

(c) *the equation $\sum_{k=0}^{n} a_k x^k = 0$ has no positive real roots.*

PROOF. Let $\phi(x) = \sum_{k=0}^{n} a_k x^k$. Let $\psi(x)$ be a polynomial, such that $\psi(x)$ and $\phi(x)\cdot\psi(x)$ have no negative coefficients. Let

$$\phi(x)\cdot\psi(x) = x^s \cdot \sigma(x),$$

where $\sigma(x)$ is a polynomial of degree m with $\sigma(0) \neq 0$. Then

$$\sigma(x)\cdot x^m \sigma\left(\frac{1}{x}\right) = \sum_{k=0}^{2m} c_k x^k$$

is a polynomial of degree $2m$ with $c_k = c_{2m-k} \geqq 0$. Put

$$\tau(x) = \psi(x) \cdot x^m \sigma\left(\frac{1}{x}\right) = \sum_r b_r x^r.$$

Then

$$\phi(x) \cdot \tau(x) = x^s \sum_{k=0}^{2m} c_k x^k,$$

where $\tau(x)$ is a polynomial with non-negative coefficients b_r and $c_k = c_{2m-k} \geqq 0$. For a convex solution $f(x)$ of (4)

$$\begin{aligned}\sum_r b_r g(x + r) &= \sum_{k=0}^{2m} c_k f(x + k + s) \\ &= \frac{1}{2} \sum_{k=0}^{2m} c_k [f(x + k + s) + f(x + 2m - k + s)] \\ &\geqq \frac{1}{2} \sum_{k=0}^{2m} c_k f(x + s + m),\end{aligned}$$

or

$$f(x + s + m) \leqq \frac{2 \sum_r b_r g(x + r)}{\sum_k c_k}.$$

As $g(x)$ is bounded above, it follows that $f(x)$ is bounded above, and hence continuous.

THEOREM 5. *If the difference equation*

$$\sum_{0=k}^{n} a_k f(x + k) = g(x) \tag{5}$$

has a continuous convex solution, and if the equation $\sum_{k=0}^{n} a_k x^k = 0$ *has a positive real root, then the difference equation has discontinuous convex solutions as well.*

PROOF. It is sufficient to prove that the equation

$$\sum_{k=0}^{n} a_k f(x + k) = 0$$

has a discontinuous convex solution, as the sum of two convex functions is again convex. Let

$$\sum_{k=0}^{n} a_k x^k = (x - \lambda) \sum_{k=0}^{n-1} b_k x^k,$$

where $\lambda > 0$. Then

$$\sum_k a_k f(x+k) = \sum_k b_k [f(x+k+1) - \lambda f(x+k)].$$

It is sufficient to show that the equation

$$f(x+1) - \lambda f(x) = 0 \tag{6}$$

has a discontinuous convex solution.

Let Σ be a basis for all real numbers; that is, every real x may be represented in one and only one way in the form

$$x = \alpha a + \beta b + \cdots + \gamma c,$$

where $a, b, \cdots, c$ are in Σ, and $\alpha, \beta, \cdots, \gamma$ are rational numbers. (The existence of such a base is proved by Hamel [2].) Without restriction of generality we may assume 1 to be an element of Σ (this comes back to assuming a normal ordering of the set of real numbers with 1 as first element). For every real x, there is then uniquely determined a number α, such that

$$x = \alpha \cdot 1 + \beta b + \cdots + \gamma c,$$

where $\alpha, \beta, \cdots, \gamma$ are rational, and $1, b, \cdots, c$ are in Σ. If $\lambda \neq 1$, define $f(x)$ by λ^α. Then

$$f(x+1) - \lambda f(x) = \lambda^{\alpha+1} - \lambda \cdot \lambda^\alpha = 0.$$

If $y = \alpha' 1 + \beta' b + \cdots + \gamma' c$ is the representation of y then

$$f\left(\frac{x+y}{2}\right) = (\lambda^\alpha \cdot \lambda^{\alpha'})^{1/2} \leqq \frac{\lambda^\alpha + \lambda^{\alpha'}}{2} = \frac{f(x) + f(y)}{2}.$$

Hence $f(x)$ is a convex solution of (6). $f(x)$ is discontinuous in every interval, as for an element $b \neq 1$ of Σ

$$f(\beta b) = 1$$

for all rational β, whereas

$$f(\alpha) = \lambda^\alpha \neq 1$$

for all rational $\alpha \neq 0$. Similarly, if $\lambda = 1$, we define $f(x) = \beta$, where β is the coefficient of the fixed element $b \neq 1$ of Σ. Then

$$f(x+1) - f(x) \equiv 0, \qquad f\left(\frac{x+y}{2}\right) \equiv \frac{f(x) + f(y)}{2},$$

$f(\alpha) = 0$ for all rational α, but $f(\beta b) = \beta \neq 0$ for all rational $\beta \neq 0$.

Note added March 15: I am indebted to Professors O. Szász and G. Szegö for the information that the Lemma had been proved by E. Meissner, Mathematische Annalen, vol. 70 (1911), pp. 223–235. See also Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 730, no. 190.

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MOMENT PROBLEM FOR A BOUNDED REGION[1]

L. B. HEDGE

1. **Introduction.** In this paper a solution of the moment problem given by Hausdorff[2] for a bounded interval is extended to any bounded region in euclidean n-space, under certain conditions on polynomial expansions over the region. The resulting solution is valid for the n-dimensional sphere, and includes the Hausdorff case as well as the known conditions on the "class" of Fourier and Fourier-Stieltjes series.[3]

2. **Definitions and notation.** Let n be a positive integer, fixed but arbitrary. R^n will denote the euclidean n-space, (x) and (y) will stand for $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$, points of R^n, and E a bounded, closed subset of R^n. ν, τ, i, j, k, and s, will be used for non-negative integers, and (k), (s), and so on, will denote ordered n-tuples of non-negative integers $(k_1, k_2, \cdots, k_n)$, $(s_1, s_2, \cdots, s_n)$, and so on, $(k)=(s)$ will mean $k_i=s_i$, $i=1, 2, \cdots, n$. (0) will mean $(0, 0, \cdots, 0)$, $\{\mu_{(m)}\}$ will be a sequence of real numbers, and $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ will be two sequences of polynomials such that

$$U_{(0)}(x) = V_{(0)}(x) = \text{const.},$$

$$\int_E U_{(k)}(x)V_{(s)}(x)dx = \begin{cases} 0, & (k) \neq (s), \\ 1, & (k) = (s), \end{cases} \tag{1}$$

and by $\int_{E(y)} f(x, y)d\Phi(E)$ will be meant the Lebesgue-Stieltjes integral over E of f considered as a function of a point (y). B will be used for any Borel set with $B \subseteq E$.

If f is integrable over E we define

$$\mathfrak{S}(f, x) \simeq \sum_{(k)} A_{(k)}V_{(k)}(x), \qquad A_{(k)} = \int_E f(x)U_{(k)}(x)dx,$$

$$S(x, y) \simeq \sum_{(k)} U_{(k)}(x)V_{(k)}(y).$$

Let L_ν for every ν be a partition of R^n into two subsets, one closed and bounded. We write $(k) \in L_\nu$ to indicate that (k) belongs to the

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[1] Presented to the Society, June 20, 1940.

[2] F. Hausdorff, *Momentprobleme für ein endliches Intervall*, Mathematische Zeitschrift, vol. 16 (1923), pp. 220–248.

[3] See, for example, A. Zygmund, *Trigonometrical Series*, Monografje Matematyczne, vol. 5, Warsaw, 1935, pp. 79–86.

bounded subset defined by L_ν, and require that for every (k) there exist a ν such that $(k)\in L_\nu$, and that $(k)\in L_\nu$ shall imply $(k)\in L_{\nu'}$ for all $\nu'\geqq\nu$. Now let

$$S_\nu(x, y) = \sum_{(k)\in L_\nu} U_{(k)}(x)V_{(k)}(y),$$

$$\mathfrak{S}_\nu(f, x) = \sum_{(k)\in L_\nu} A_{(k)}V_{(k)}(x) = \int_E S_\nu(x, y)f(y)dy.$$

If $T: \|a_{ij}\|$ is any regular Toeplitz transformation,[4] we write

$$T\mathfrak{S}_\nu(f, x) = \int_E TS_\nu(x, y)f(y)dy = \int_E K_\nu(x, y)f(y)dy.$$

If P is a polynomial in (x) we denote by $\mu_{(x)}(P)$ the expression resulting from the substitution of $\mu_{m_1,m_2,\cdots,m_n}$ for $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ in P.

3. **Moment problem.** A solution of the moment problem for the set E is given in the following theorem:

THEOREM. *Given* $\{U_{(k)}(x)\}$, $\{V_{(k)}(x)\}$, $\{L_\nu\}$, *and* T *satisfying the conditions above, and such that* $TS_\nu(x, y)=K_\nu(x, y)\geqq 0$ *for all* (x), $(y)\in E$, *and all* ν, *and such that for any* f *integrable over* E $T\mathfrak{S}_\nu(f, x)\to f(x)$ *for every* $(x)\in E$ *for which* f *is continuous, and uniformly on* E *if* f *is continuous on* E, *then in order that a sequence* $\{\mu_{(m)}\}$ *be expressible in the form*

$$\mu_{m_1,m_2,\cdots,m_n} = \int_E x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}d\Phi(E),$$

where Φ *is completely additive, defined over at least all Borel sets of* R^n, *and with*

(1) $\int_E |d\Phi(E)| \leqq M$,
(2) $\Phi(B)\geqq 0$,
(3) $\Phi(B)=\int_B\phi(x)dx$ *and with*
(3a) $\phi\in L_E^p$, $p>1$,
(3b) $\phi\in L_E$,
(3c) $|\phi|\leqq M$,
(3d) $\phi\in C_E$,

it is necessary and sufficient that

(1) $\int_E |\mu_{(y)}\{K_\nu(x, y)\}|dx\leqq M$ *for all* ν,
(2) $\mu_{(y)}\{K_\nu(x, y)\}\geqq 0$ *for all* $(x)\in E$ *and all* ν,
(3a) $\int_E |\mu_{(y)}\{K_\nu(x, y)\}|^p dx\leqq M$ *for all* ν,

[4] Zygmund, loc. cit., pp. 40–43.

(3b) $\lim_{\nu,\tau\to\infty}\int_E|\mu_{(y)}\{K_\nu(x,y)\}-\mu_{(y)}\{K_\tau(x,y)\}|\,dx=0$,
(3c) $|\mu_{(y)}\{K_\nu(x,y)\}|\leqq M$ *for all* $(x)\in E$ *and all* ν,
(3d) $\lim_{\nu,\tau\to\infty}|\mu_{(y)}\{K_\nu(x,\ y)\}-\mu_{(y)}\{K_\tau(x,\ y)\}|=0$ *uniformly in* $(x)\in E$.

The proof in each of the six cases closely parallels that of Hausdorff.[2] The proof is given for case (1) to indicate the modifications:

Necessity. We have

$$|\mu_{(y)}\{K_\nu(x,y)\}| = \left|\int_{E(y)} K_\nu(x,y)\,d\Phi(E)\right|$$

$$\leqq \int_{E(y)} K_\nu(x,y)\,|\,d\Phi(E)\,|,$$

$$\int_E|\mu_{(y)}\{K_\nu(x,y)\}|\,dx \leqq \int_E\left\{\int_E K_\nu(x,y)dx\right\}|\,d\Phi(E)\,|$$

$$\leqq C\int_E|\,d\Phi(E)\,|\leqq M$$

for

$$K_\nu(x,\ y)=\sum_{j=0}^{\infty}a_{\nu j}\sum_{(k)\,\epsilon\,L_j}U_{(k)}(x)V_{(k)}(y)$$

$$\int_E K_\nu(x,\ y)dx=\sum_{j=0}^{\infty}a_{\nu j}\sum_{(k)\,\epsilon\,L_j}V_{(k)}(y)\int_E U_{(k)}(x)dx$$

$$=\sum_{j=0}^{\infty}a_{\nu j}\leqq\sum_{j=0}^{\infty}|\,a_{\nu j}|\leqq C.$$

Sufficiency. Let

$$\Phi_\nu(B)=\int_B\mu_{(y)}\{K_\nu(x,\ y)\}dx,$$

$$\int_E|\,d\Phi_\nu(E)\,|=\int_E|\,\mu_{(y)}\{K_\nu(x,\ y)\}\,|\,dx\leqq M$$

and, by a well known theorem of Helly, there is a subsequence $\{\Phi_{\nu'}\}$ and a function Φ such that $\int_E|\,d\Phi(E)\,|\leqq M$ and $\Phi_{\nu'}(B)\to\Phi(B)$, and also $\int_E V_{(k)}(y)d\Phi_{\nu'}(E)\to\int_E V_{(k)}(y)d\Phi(E)$ whence $\mu_{(y)}\{V_{(k)}(y)\}=\int_E V_{(k)}(y)d\Phi(E)$, and Φ is a solution.

4. **Examples and conclusion.** If E is the unit sphere in R^n, $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ may be taken as the normalized polynomials of Appell-

Didon,[5] $(k)\in L$, to mean $\sum_{i=1}^{n} k_i \leqq \nu$, and T any (C, r) with $r \geqq n+1$.[6] In particular, for $n=1$ this reduces to the Hausdorff solution for the unit interval. If E is the circumference of the unit circle we may set $U_0(x) = V_0(x) = (2\pi)^{-1/2}$, and, for $k>0$,

$$U_{2k}(x) = V_{2k}(x) = (\pi)^{-1/2} \cos k\theta, \quad U_{2k-1}(x) = V_{2k-1}(x) = (\pi)^{-1/2} \sin k\theta$$

with $(s)\in L$, meaning $s \leqq 2\nu$, T any (C, r) with $r \geqq 1$.[7] Sequences $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ can be constructed by the Schmidt process for any bounded region in R^n. It would be interesting to know whether regular Toeplitz transformations of the type required for the present theorem exist in general.

BROWN UNIVERSITY

[5] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques; Polynomes d'Hermite*, Paris, 1926.

[6] L. Koschmieder, *Über die C-Summierbarkeit gewisser Reihen von Didon und Appell*, Mathematische Annalen, vol. 104 (1931), pp. 387–402.

[7] L. Fejer's theorem. See, for instance, Zygmund, loc. cit., p. 45.

NOTE ON AN INTEGRAL OF BIERENS DE HAAN

THOMAS JAMES HIGGINS

In connection with certain potential problems, the author has recently had occasion to evaluate integrals of the form

$$I_n = \int_0^\infty \frac{\epsilon^{-px}}{x^{n+1}} \sin q_0 x \cdot \sin q_1 x \cdot \sin q_2 x \cdots \sin q_n x \, dx.$$

In Table 371 of Bierens de Haan's *Nouvelles Tables d'Intégrales Définies* a rule is given for evaluation of this integral. An attempt to use it led to finding that it was incorrect.

This integral was originally considered by de Haan in a paper published in Dutch in the rather inaccessible Archief, Wiskundig Genootschap onder de Zinspreuk, part 1, 1856–1859, pp. 288–315. Reference to this article revealed that while de Haan did not give a specific rule for the above integral, he did discuss it in general. Evidently, in the compilation of these tables in French a garbled version of this discussion was inserted. As integrals of this form occur in several potential problems of current interest,[1] it is thought worthwhile to point out this error in these universally used tables and to give the following rule, ascertained partly from de Haan's discussion and partly from direct inspection of a number of specific cases given in the same article. The correct formula is:

$$I = \frac{1}{n!2^n} \sum_{j=1}^{j=2^n} (-1)^{r_j} \left[T_j \arctan \frac{k_j}{p} - L_j \log (k_j^2 + p^2)^{1/2} \right],$$

where

(a) k_j is any one of the 2^n possible sums obtainable from $(q_0 \pm q_1 \pm q_2 \pm \cdots \pm q_n)$ by selection of the $+$ and $-$ signs;

(b) r_j is the number of negative signs in k_j;

(c) T_j is the sum of the 1st, 3rd, 5th, $\cdots$ terms of

$$k_j^n + C_{n,n-1} p k_j^{n-1} - C_{n,n-2} p^2 k_j^{n-2} - C_{n,n-3} p^3 k_j^{n-3} + C_{n,n-4} p^4 k_j^{n-4} + C_{n,n-5} p^5 k_j^{n-5} - C_{n,n-6} p^6 k_j^{n-6} - \cdots p^n;$$

[1] The author encountered these integrals in seeking to determine the vector potential of two parallel, infinitely long, tubular rectangular conductors carrying currents in opposite directions. In general, they arise in two dimensional problems, where it is necessary to obtain a potential satisfying Poisson's equation over certain rectangular areas and Laplace's equation over the remainder of space.

(d) L_j is the sum of the remaining terms.

As an illustration we evaluate the intégral

$$I_3 = \int_0^\infty \frac{e^{-px}}{x^3} \sin^2 ax \cdot \sin bx \, dx.$$

Now

$$k_1 = a + a + b = 2a + b; \quad (-1)^{r_1} = 1;$$
$$k_2 = a + a - b = 2a - b; \quad (-1)^{r_2} = -1;$$
$$k_3 = a - a + b = b; \quad (-1)^{r_3} = -1;$$
$$k_4 = a - a - b = -b; \quad (-1)^{r_4} = 1;$$

and from

$$k_j^2 + C_{2,1} p k_j - p^2,$$

we have

$$T_j = k_j^2 - p^2, \qquad L_j = 2pk_j.$$

Hence

$$I_3 = \frac{1}{2^2 2!} \sum_{j=1}^{j=4} (-1)^{r_j} \left[(k_j^2 - p^2) \arctan \frac{k_j}{p} - 2pk_j \log (k_j^2 + p^2)^{1/2} \right].$$

This confirms the integral #2, Table 370.

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ELEMENTARY PROOF OF A THEOREM ON LORENTZ MATRICES

E. R. VAN KAMPEN

Let x and y be real n- and m-vectors and x^2, y^2 the scalar squares of x, y. The corresponding Lorentz matrices are matrices of $(n+m)$-dimensional real linear transformations which leave the quadratic form x^2-y^2 invariant. Let the transformation be written in the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax+By \\ Cx+Dy \end{pmatrix}. \tag{1}$$

Then *the signs of the determinants* $|A|$ *and* $|D|$ *form two* 1*-dimensional representations of the Lorentz group*. Two algebraic proofs at present available for this fact[1] depend on a recursive factorization of the Lorentz matrix into simple factors or on deeper facts from the theory of representations. On the other hand, a simple topological proof may be given in quite an obvious manner. In this note the topological proof is briefly sketched and then a simple algebraic proof is given which does not depend on recursive factorization or representation theory and is valid in any real field.

The set defined by $x^2-y^2 \geqq 1$ in the real $(n+m)$-dimensional space possesses one basic $(n-1)$-dimensional (finite) cycle Γ which can most easily be represented by the $(n-1)$-dimensional basic cycle of the $(n-1)$-dimensional sphere $x^2=1$, $y=0$. Now Γ is transformed by (1) into a cycle homologous to $+\Gamma$ or to $-\Gamma$ according as $|A|$ is positive or negative. The formal proofs of these topological facts are obtained most easily from the remark that the whole space $x^2-y^2 \geqq 1$ can be retracted into its subset $x^2=1$, $y=0$ by a deformation which does not change the value of $x/(x^2)^{1/2}$ for any point. That *sign* $|A|$ is a one-dimensional representation of the Lorentz group is of course evident from the fact that Γ is transformed by (1) into a cycle homologous to *sign* $|A| \cdot \Gamma$. The statement concerning the signature of $|D|$ depends on a similar consideration for the set defined by $x^2-y^2 \leqq -1$.

Now let the elements of the matrix in (1) belong to any real field. Let the unit matrices of dimensions n and m be denoted by E_n and E_m. The fact that the matrix in (1) is a Lorentz matrix may be expressed by the relations:

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[1] Cf. W. Givens, *Factorization and signatures of Lorentz matrices*, this Bulletin, vol. 46 (1940), pp. 81–85, where other references are given. My thanks are due to Dr. Murnaghan who drew my attention to the above theorem.

$$(2) \qquad A'A - C'C = E_n, \qquad D'D - B'B = E_m, \qquad A'B = C'D,$$

which may be obtained by forming the expression x^2-y^2 for the vector on the right in (1).

If P is a matrix of m rows and n columns, such that $E_n-P'P$ is positive definite, then the sign of the determinant $|A+BP|$ is independent of P; in particular $|A+BP| \neq 0$ and $|A| \neq 0$.

In fact, from (2) one easily obtains the identity

$$(A + BP)'(A + BP) = (C + DP)'(C + DP) + E_n - P'P.$$

Since $E_n-P'P$ is assumed to be positive definite, this implies that $(A+BP)'(A+BP)$ is positive definite. Thus $|A+BP| \neq 0$ and, by choosing $P=0$, also $|A| \neq 0$. On replacing P by tP, one sees that the determinant $|A+tBP|$, which is a polynomial in the parameter t, is never 0 while $-1 \leqq t \leqq 1$. For $E_n-t^2P'P=E_n-P'P+(1-t^2)P'P$ is positive definite if $-1 \leqq t \leqq 1$. Thus the polynomial $|A+tBP|$ cannot change its sign as t varies between 0 and 1. In the field of real numbers this is evident. If the underlying field is any real field, and if the polynomial $|A+tBP|$ took both possible signs for $-1 \leqq t \leqq 1$, then one could adjoin to the field a root of $|A+tBP|=0$ which lies between -1 and 1. In the enlarged field one obtains of course a contradiction with the fact that $|A+tBP| \neq 0$ for $-1 \leqq t \leqq 1$.

Let the product of two Lorentz matrices be written in the form

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix}.$$

Then one has

$$A_1A_2 + B_1C_2 = (A_1 + B_1C_2A_2^{-1})A_2 = (A_1 + B_1P)A_2,$$

where $P = C_2A_2^{-1}$. But

$$\begin{aligned} E_n - P'P &= E_n - (A_2')^{-1}C_2'C_2A_2^{-1} = (A_2')^{-1}(A_2'A_2 - C_2'C_2)A_2^{-1} \\ &= (A_2')^{-1}A_2^{-1} \end{aligned}$$

is a positive definite matrix, so that sign $|A_1+B_1C_2A_2^{-1}| = \text{sign } |A_1+B_1P| = \text{sign } |A_1|$. Thus

$$\text{sign } |A_1A_2 + B_1C_2| = \text{sign } |A_1| \cdot \text{sign } |A_2|.$$

This completes the algebraic proof of the above theorem.

The geometrical content of the proof becomes clearer, if one realizes that the n-dimensional linear spaces with the equations $y=Px$ (where $E_n-P'P$ is positive definite) are precisely those spaces through the origin which meet the quadratic $x^2-y^2=1$ in a completely elliptical

quadratic (and the cone $x^2-y^2=0$ in its vertex only). Thus this system of linear spaces is invariant under the Lorentz group. That the sign of $|A+BP|$ is independent of P means that the orientation of all spaces $y=Px$ is left invariant by a Lorentz matrix with $|A|>0$ and is changed into its opposite by a Lorentz matrix with $|A|<0$. Complications in preceding proofs of the theorem apparently originate either from the inclusion of the proof that every matrix P with positive definite $E-P'P$ is the matrix CA^{-1} of a Lorentz transformation (1) and/or of the proof that the subgroup of the Lorentz group defined by $|A|>0$, $|D|>0$ is connected.

The 1-dimensional representation of the Lorentz group given by the determinants of the Lorentz matrix (1) *is the product of the two representations given by the signs of* $|A|$ *and* $|D|$. In fact, D as well as A is nonsingular. Thus[2]

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot \begin{vmatrix} A & B \\ D^{-1}C & E_m \end{vmatrix} = |D| \cdot \begin{vmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & E_m \end{vmatrix}$$
$$= |D| \cdot |A - BD^{-1}C|,$$

so that, since $BD^{-1}=A'^{-1}C'$ and $|A'|=|A|$,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot |A - A'^{-1}C'C| = \frac{|D|}{|A|} |A'A - C'C| = \frac{|D|}{|A|}.$$

Thus the sign of

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

is the product of the signs of $|D|$ and $|A|$. Since

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{|A|}{|D|}$$

may be similarly proved, one has

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \text{sign } |A| \cdot \text{sign } |D|.$$

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[2] Cf. J. Williamson, *The expansion of determinants of composite order*, American Mathematical Monthly, vol. 40 (1933), pp. 65–69.

ERRORS IN HAYASHI'S TABLE OF BESSEL FUNCTIONS FOR COMPLEX ARGUMENTS[1]

ARNOLD N. LOWAN AND GERTRUDE BLANCH[2]

The Project for the Computation of Mathematical Tables is engaged in the computation of an extensive table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for complex arguments.[3] Certain formulas developed for the purpose of checking the tables could also be applied to checking systematically the above-mentioned table of Hayashi (credited to A. Dinnik) and it was deemed worth while to do so.

All values on the rays $\phi=\frac{1}{2}\pi$ and $\phi=\frac{1}{4}\pi$ were checked against the 15-decimal place values computed by us. Those on the ray $\phi=\frac{1}{2}\pi-0.001$ were differenced as a function of r; one error was discovered and corrected.

The entries on all the other rays were differenced as a function of r. In addition, "summation" tests were applied, based on the following theorem due to G. Blanch:

For a fixed r, let $U_0(r, \phi_p)$ represent the real component of $J_0(r, \phi_p)$, where $\phi_p=p\pi/16$, and p ranges from 0 to 7. Similarly, let $V_0(r, \phi_p)$, $U_1(r, \phi_p)$ represent the corresponding values of these functions for a fixed r, and ϕ ranging as before. Then the following formulas hold:

$$(1) \qquad \sum_{p=0}^{7} U_0(r, \phi_p) = 8 + \tfrac{1}{2}[J_0(r) - I_0(r)] + R_1,$$

$$(2) \qquad \sum_{p=1}^{7} V_0(r, \phi_p) \sin(2\phi_p) = -r^2 + R_2,$$

$$(3) \qquad \sum_{p=1}^{7} U_1(r, \phi_p) \cos\phi_p = 2r - \tfrac{1}{2}J_1(r) + R_3,$$

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[1] Published in *Fünfstellige Funktionentafeln*, by K. Hayashi, Berlin, Springer, 1930, pp. 105–109 (from Prof. A. Dinnik, Yekaterinoslaw, 1922). Tabulated in the form $J_0(z)=U_0+iV_0$; $J_1(z)=U_1+iV_1$ for $z=r\exp(i\phi)$. Range of r: 0(0.2)8. Range of ϕ: $0(\pi/16)\pi/2$ for U_0 and V_1; $\pi/16(\pi/16)7\pi/16$ and $\pi/2-0.001$ for V_0 and U_1. To 4 decimal places.

[2] The results reported in this paper were obtained in the course of work done by the Project for the Computation of Mathematical Tables, O.P. No. 65-2-97-33, Work Projects Administration, New York, N. Y., operated under the sponsorship of Dr. Lyman J. Briggs, Director of the National Bureau of Standards. The authors wish to express their appreciation to the W.P.A. and to the Sponsor of this Project for permission to publish these results.

[3] This work has since been completed.

$$(4) \qquad \sum_{p=1}^{7} V_1(r, \phi_p) \sin \phi_p = 2r - \tfrac{1}{2} I_1(r) + R_4,$$

where $|R_i| < 7 \times 10^{-7}$ *for* $r \leqq 8$, $i = 1, 2, 3, 4$.

The summations indicated above were actually performed, and wherever the discrepancy between the theoretical and actual sums exceeded one unit in the fourth decimal place, the source of a possible error was sought, along rays where the differences were irregular. Since each sum contained a group of 7 rounded entries, an occasional discrepancy of two units was to be expected. However, in such cases, the entries giving rise to the discrepancy were checked with special precaution, especially in formulas (2), (3), and (4), where multiplication of the entries by $\sin \phi_p$ or $\cos \phi_p$ might possibly mask an error larger than that shown by the discrepancy.

Because of the great number of errors in the table, it would have been very difficult to localize them by differencing methods alone; but with the aid of the summation tests, the erroneous entries were picked out systematically. The values in error were then re-computed. After all errors were corrected, the differences along the rays were sufficiently regular, and the discrepancies in the summation tests were no larger than 2 units in the fourth decimal place anywhere.

It was obviously impossible to check last-place errors of about a unit, except along the rays which we ourselves have computed. A number of additional errors of one unit were discovered during the recomputation of entries on complementary rays. [Thus, for example, $U_0(r, \phi) = A + B$, where $A = \sum_{k=0}^{\infty} -r^{4k+2} \cos (4k+2)\phi/(k!)^2$, and $B = \sum_{k=0}^{\infty} r^{4k} \cos 4k\phi/(k!)^2$; and $-A + B = U_0(r, \frac{1}{2}\pi - \phi)$. Hence one set of values of A and B yields the entries on two (complementary) rays.] However, it is not deemed worth while to report these errors of one unit. Readers interested in obtaining a list of those which came to light might communicate with the authors. Errors of more than one unit are given below:

ERRORS IN HAYASHI'S TABLE OF BESSEL FUNCTIONS FOR COMPLEX ARGUMENTS

		U_0				V_0	
		Hayashi's	*Correct*			*Hayashi's*	*Correct*
r	ϕ	*Value*	*Value*	r	ϕ	*Value*	*Value*
6.8	$\pi/16$	0.5338	0.5837	1.0	$\pi/16$	−0.0880	−0.0850
5.2	$2\pi/16$	−0.6625	−0.6601				
7.4	$2\pi/16$	0.5079	2.5079				
1.4	$3\pi/16$	0.7781	0.7731	4.6	$3\pi/16$	0.4125	0.4095
5.6	$3\pi/16$	−2.0436	−2.0636	6.0	$3\pi/16$	4.4654	4.4854

		U_0				V_0	
		Hayashi's	*Correct*			*Hayashi's*	*Correct*
r	ϕ	*Value*	*Value*	r	ϕ	*Value*	*Value*
5.8	$3\pi/16$	−1.6127	−1.6428				
6.8	$3\pi/16$	2.4803	2.8403				
5.4	$4\pi/16$	−7.6676	−7.6674	6.8	$4\pi/16$	18.0734	18.0736
				7.4	$4\pi/16$	28.6088	27.6088
				8.0	$4\pi/16$	35.0280	35.0167
3.8	$5\pi/16$	−0.9092	−1.0874	0.8	$5\pi/16$	−0.1528	−0.1523
5.0	$5\pi/16$	−8.9930	−9.0929	6.0	$5\pi/16$	−2.8406	−2.8206
6.2	$5\pi/16$	−26.2519	−28.2518				
				6.8	$6\pi/16$	−56.4404	−56.4402
				5.8	$7\pi/16$	−40.0003	−42.8973
5.2	$8\pi/16$	32.5636	32.5836	5.6	$7\pi/16$	−35.0061	−35.0360
6.8	$8\pi/16$	140.1364	140.1362				
8.0	$8\pi/16$	427.5529	427.5641				
				8.0	$\pi/2-0.001$	−3.3587	−3.1989
		U_1				V_1	
0.2	$\pi/16$	0.0966	0.0977	5.6	$\pi/16$	0.0118	0.0818
5.2	$\pi/16$	−0.5248	−0.5255				
				2.2	$2\pi/16$	−0.0745	−0.0754
				8.0	$2\pi/16$	2.6334	2.5000
7.6	$3\pi/16$	−3.8432	−3.8428	5.0	$3\pi/16$	−2.2772	−2.2792
7.8	$3\pi/16$	−2.5846	−2.5841	5.6	$3\pi/16$	−1.7016	−1.7014
8.0	$3\pi/16$	−0.9458	−0.9448	7.4	$3\pi/16$	8.6155	7.3091
				7.8	$3\pi/16$	10.2946	10.2950
				1.0	$4\pi/16$	0.3005	0.3076
5.6	$5\pi/16$	4.6486	4.6489	2.6	$5\pi/16$	0.5418	0.5412
7.4	$5\pi/16$	−44.1809	−42.8744	6.6	$5\pi/16$	−34.4889	−34.4849
				7.6	$5\pi/16$	−52.7084	−52.7088
				7.8	$5\pi/16$	−53.9269	−53.9275
2.6	$6\pi/16$	1.8123	1.8133	8.0	$5\pi/16$	−53.3290	−53.3300
6.2	$6\pi/16$	37.4187	37.4126				
8.0	$6\pi/16$	55.4879	55.3546				
1.4	$7\pi/16$	0.2241	0.2441	0.2	$7\pi/16$	0.0945	0.0985
				2.8	$7\pi/16$	1.7755	2.7755
				5.2	$7\pi/16$	15.7771	15.7764

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RECURRENCE OF SYMBOLIC ELEMENTS IN DYNAMICS[1]

RUFUS OLDENBURGER

1. **Introduction.** Morse and Hedlund [1] have given a symbolic treatment of modern theoretical dynamics as developed by Birkhoff [2] and others. In the Morse-Hedlund viewpoint the concept of recurrence plays an important role. To establish various theorems on symbolic trajectories Morse and Hedlund introduced "symbolic elements," the analogues of line elements on ordinary trajectories, a symbolic element being the notion of a trajectory T and a particular symbol in T. In the present paper we shall be concerned primarily with the question: "*How are recurrence of a trajectory T and elements based on T related?*"

2. **Definitions.** For terms defined elsewhere and used here the reader is referred to other papers [1, 3]. Let rays R_i $(i=1, 2, 3, 4)$ be given by

$$a_{i1}\, a_{i2}\, a_{i3} \cdots .$$

The distance E_1E_2 between the elements $E_1=(R_1, R_2)$ and $E_2=(R_3, R_4)$ is defined to be $1/n$ where n is such that $a_{1j}=a_{3j}$, $a_{2j}=a_{4j}$ for each value of j in the range $1, 2, \cdots, n$, while

$$(a_{1,n+1}, a_{2,n+1}) \neq (a_{3,n+1}, a_{4,n+1}).$$

The element E_1 is the ray

$$A_1A_2A_3 \cdots , \tag{1}$$

where A_j denotes the pair of symbols (a_{1j}, a_{2j}). In what follows the term "recurrence of E_1" designates the recurrence of the ray (1). We recall that a ray (1) is *recurrent* if for each n there is an m such that each n-block in (1) is contained in each m-block of (1). For each n the value $R(n)$ of the *recurrency* function of (1) is the smallest m with the property just mentioned.

3. **Recurrence.** It is evident that the recurrence of a single element based on a trajectory T does not imply the recurrence of T.

THEOREM 1. *If each element based on a trajectory T is recurrent, T is recurrent.*

The recurrence of each ray based on T obviously implies the recur-

[1] Presented to the Society, December 28, 1939, under the title *Symbolic elements in dynamics.*

rence of T. The recurrence of each element based on T implies the recurrence of each ray based on T, whence T is recurrent.

THEOREM 2. *There exists a recurrent trajectory with a non-recurrent element.*

If α is irrational, the trajectory $T(0, \alpha)$ of Morse and Hedlund [4] is recurrent and is of the form $R^{-1}abR$. The element (bR, aR) is not recurrent.

The proof of the theorem to follow is simple and will be omitted.

THEOREM 3. *A trajectory T is identical with its inverse if and only if $T=R^{-1}R$ or $T=R^{-1}aR$ for some ray R and symbol a.*

THEOREM 4. *If a trajectory T is recurrent and is identical with its inverse, each element based on T is recurrent.*

We assume first that $T=R^{-1}R$ for some ray R. Consider an element $E=(B^{-1}R, R_1)$, where B is such that $R=BR_1$. We permit B to be vacuous, in which case $E=(R, R_1)$. Let G denote the leading s-block of R_1 for a given value of s, and let r denote the length of B. Let J denote the leading s-block of $B^{-1}R$. Then J in $B^{-1}R$ matches G in R_1, and the pair of blocks J, G corresponds to the block $A_1 \cdots A_s$ in the representation (1) of E. The block $G^{-1}B^{-1}BG$ is a subblock of T. Let $R(n)$ denote the recurrency function of T, whence $G^{-1}B^{-1}BG$ occurs in each $R(2r+2s)$-block of T. Let n be the number of symbols preceding one such block $G^{-1}B^{-1}BG$ in R, so that

$$R = HG^{-1}B^{-1}BGR_2,$$

where H is a block of length n and R_2 is a subray of R. For brevity we introduce the notation $K=G^{-1}B^{-1}BG$ so that $R=HKR_2$. It is no restriction on the generality of the argument to follow to suppose that $n \geqq r$ so that the block K is a subblock of R_1. The final block G in K is preceded in R_1 by a subblock of length $n+r+s$. In the ray $B^{-1}R$ the final subblock $B^{-1}BG$ of K is preceded by a block of length $n+r+s$. Thus the initial s-block J of the subblock $B^{-1}BG$ of K in $B^{-1}R$ matches the final block G of K in R_1. We have proved that in each $R(2r+2s)$-block of R_1 there is a subblock G matching a subblock J of $B^{-1}R$ whence each $R(2r+2s)$-block in the representation (1) contains a block identical with $A_1 \cdots A_s$. Since each block in (1) is contained in the block $A_1 \cdots A_s$ for large enough s, it follows that E is recurrent.

We suppose, finally, that $T=R^{-1}aR$ for some ray R and symbol a. Let r and s be arbitrary non-negative integers. We consider an ele-

ment $E=(D^{-1}aR,\ R_1)$ where D is the leading r-block of R so that $R=DR_1$, R_1 being a subray of R. We let G denote the leading s-block of R_1 so that $R_1=GR_2$ where R_2 is a subray of R_1. Evidently, T contains the subblock $G^{-1}D^{-1}aDG$ and the initial block G of R_1 matches the leading s-block J of $D^{-1}aR$. The last block G in any block $G^{-1}D^{-1}aDG$ of R_1 matches the leading block J of a subblock $D^{-1}aDG$ in $D^{-1}aR$, whence E is recurrent.

By Theorem 3 the above argument proves that each element E based on T is recurrent.

THEOREM 5. *If for some block C and ray R a trajectory T is of the form $R^{-1}CR$, and if each element based on T is recurrent, the trajectory T is identical with its inverse.*

We suppose first that C is of even length $2q$, and we accordingly write $C=C_1C_2$ where C_1 and C_2 are of length q. We consider the element $E=(C_1^{-1}R,\ C_2R)$ based on T. Since E is recurrent, C_2 occurs in the subray R matched in C_2R by a block C_1^{-1} in $C_1^{-1}R$. It follows that C_1^{-1} matches C_2 in R, whence $C_1=C_2^{-1}$ and $T=T^{-1}$.

If now C is of odd length $2q+1$, we write $C=C_1aC_3$ where C_1 and C_3 are of length q. Let the initial symbol of R be denoted by b. Since the element $E=(C_1^{-1}R,\ aC_3R)$ is recurrent, there is a subblock aC_3 of R matched in aC_3R by a block $C_1^{-1}b$ in the ray $C_1^{-1}R$. Hence there is a block C_3 in R matching a block C_1^{-1} in R. Thus $C_1=C_3^{-1}$ and $T=T^{-1}$.

The Morse recurrent trajectory T extensively studied in the literature is a trajectory identical with its inverse, whence each element based on T is recurrent. Morse and Hedlund gave methods termed projection, reduction, association, and substitution by which one can obtain a recurrent trajectory from a given recurrent trajectory. One can prove readily that the elements based on the trajectories obtained by derivation from the Morse recurrent trajectory are recurrent.

Elsewhere the author proved (3) that the exponent trajectory T_e of a recurrent trajectory T is generated by a finite number of symbols, and is recurrent. Let $T_1,\ T_2,\ \cdots$ be a sequence of trajectories such that T_i is the exponent trajectory of T_{i-1} for each i. We term $T_2,\ T_3,\ \cdots$ the *successive exponent trajectories* of T. One can prove readily that the successive exponent trajectories of the Morse recurrent trajectory are each of the form $R^{-1}aR$ where a is a symbol and R is a ray, whence the elements based on these successive trajectories are recurrent.

We let $R_i(n)$ denote the recurrency function of the element E_i based on a trajectory T. If for each i the set $R_i(1),\ R_i(2),\ \cdots$ is bounded, T is said to be *E-recurrent*. For the Morse recurrent trajectory one

can prove that the set $R_i(1)$ is not bounded whence this trajectory is not E-recurrent. The existence of a nonperiodic E-recurrent trajectory is still an open question.

References

1. M. Morse and G. A. Hedlund, *Symbolic dynamics*, American Journal of Mathematics, vol. 60 (1938), pp. 815–866.

2. G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, vol. 9, New York, 1927.

3. R. Oldenburger, *Exponent trajectories in symbolic dynamics*, Transactions of this Society, vol. 46 (1939), pp. 453–466.

4. Morse and Hedlund, *Symbolic dynamics*, II. *Sturmian trajectories*, American Journal of Mathematics, vol. 52 (1940), pp. 1–42.

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COMMENTS ON CANONICAL LINES

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1. **Introduction.** In this paper we propose to find the equations of the canonical edges of Green using a conjugate net as the parametric net of an analytic surface; to give a new interpretation to the line called by Davis the associate line of collineation; and, finally, to make a generalization of the canonical quadric of Davis.

2. **Analytic basis.** Let the projective homogeneous coordinates $x^1, \cdots, x^4$ of a point P_x on a surface S in ordinary space be analytic functions of two independent variables u, v. In the notation[1] of Lane if the parametric curves on S form a conjugate net, the coordinates x of the point P_x and the coordinates y of a point which is the harmonic conjugate of the point P_x with respect to the foci of the axis of the point P_x, satisfy a system of equations of the form

$$\begin{aligned} x_{uu} &= px + \alpha x_u + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \delta x_v + Ny, \qquad LN \neq 0. \end{aligned} \tag{1}$$

The ray-points of the net at the point P_x are given by the formulas

$$x_{-1} = x_u - bx, \qquad x_1 = x_v - ax.$$

Some of the invariants of the net are

$$\begin{aligned} H &= c + ab - a_u, & K &= c + ab - b_v, \\ \mathfrak{H} &= c + ab + b_v - \delta_u, & \mathfrak{K} &= c + ab + a_u - \alpha_v, \\ 8\mathfrak{B}' &= 4a - 2\delta + (\log r)_v, & r &= N/L, \\ 8\mathfrak{C}' &= 4b - 2\alpha - (\log r)_u. \end{aligned} \tag{2}$$

If the covariant tetrahedron, x, x_1, x_{-1}, y is used as the local tetrahedron of reference, a power series expansion[2] for one nonhomogeneous coordinate z of a point on the surface in terms of the other two coordinates x, y is

$$\begin{aligned} z = {} & \tfrac{1}{2}(Lx^2 + Ny^2) + \tfrac{4}{3}(L\mathfrak{C}'x^3 + N\mathfrak{B}y^3) + c_0x^4 \\ & + 4c_1x^3y + 4c_3xy^3 + c_4y^4 + \cdots, \end{aligned} \tag{3}$$

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[1] Lane, *Conjugate nets and the lines of curvature*, American Journal of Mathematics, vol. 53 (1931), p. 574.

[2] Lane, *A canonical power series expansion for a surface*, Transactions of this Society, vol. 37 (1935), p. 481.

where the coefficients c_0, c_1, c_3, c_4 are given by

$$c_0 = \tfrac{1}{3}L\mathfrak{C}'[12\mathfrak{C}' + (\log \mathfrak{C}'r^{1/2})_u], \qquad 4c_1 = \tfrac{1}{6}L(H - \mathfrak{H}),$$
$$c_4 = \tfrac{1}{3}N\mathfrak{B}'[12\mathfrak{B}' + (\log \mathfrak{B}'r^{-1/2})_v], \quad 4c_3 = \tfrac{1}{6}N(K - \mathfrak{K}),$$

and the coefficient of x^2y^2 is zero.

We will assume that the surface S is non-ruled,[3] that is, that $\mathfrak{B}'^2 + r\mathfrak{C}'^2 \neq 0$.

3. **The second canonical edge of Green.** Segre has shown[4] that the second[5] edge of Green is the line of inflexion of the nodal cubic in which the tangent plane at a point P_x of an analytic surface S intersects the four-parameter family of nondegenerate algebraic cubic surfaces having fourth order contact with S at P_x.

The equation in nonhomogeneous coordinates of a general algebraic cubic surface through the point P_x is

$$\begin{aligned} &a_1x + a_2y + a_3z + b_1x^2 + b_2y^2 + b_3z^2 + b_4xy + b_5xz \\ &\quad + b_6yz + d_1x^3 + d_2y^3 + d_3z^3 + d_4x^2y + d_5x^2z + d_6xy^2 \\ &\quad + d_7y^2z + d_8xz^2 + d_9yz^2 + d_{10}xyz = 0. \end{aligned} \tag{4}$$

If this surface has fourth order contact with the surface S at the point P_x, its coefficients satisfy the following conditions:

$$\begin{aligned} &a_1 = a_2 = b_4 = 0, \qquad a_3L + 2b_1 = 0, \qquad a_3N + 2b_2 = 0, \\ &8L\mathfrak{C}'a_3 + 3Lb_5 + d_1 = 0, \qquad Lb_6 + 2d_4 = 0, \\ &8N\mathfrak{B}'a_3 + 3Nb_6 + d_2 = 0, \qquad Nb_5 + 2d_6 = 0, \\ &12c_0a_3 + 3L^2b_3 + 16L\mathfrak{C}'b_5 + 6Ld_5 = 0, \\ &12c_4a_3 + 3N^2b_3 + 16N\mathfrak{B}'b_6 + 6Nd_7 = 0, \\ &12c_1a_3 + 8L\mathfrak{C}'b_6 + 3Ld_{10} = 0, \\ &12c_3a_3 + 8N\mathfrak{B}'b_5 + 3Nd_{10} = 0, \\ &LNb_3 + Nd_5 + Ld_7 = 0. \end{aligned} \tag{5}$$

The tangent plane, $z=0$, intersects this cubic surface in the cubic curve

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[3] Green, *Projective differential geometry of one-parameter families of curves and conjugate nets on a curved surface*, First Memoir, American Journal of Mathematics, vol. 37 (1915), p. 239.

[4] B. Segre, *La cubique indicatrice de l'élément linéaire projectif d'une surface*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 184, p. 729.

[5] This classification of the edges of Green is the reverse of that introduced by Green.

$$z = b_1x^2 + b_2y^2 + d_1x^3 + d_2y^3 + d_4x^2y + d_6xy^2 = 0, \tag{6}$$

where the coefficients b_1, b_2, d_1, d_2, d_4, d_6 can be expressed in terms of the invariants given by equations (2) and a_3.

Following the usual method of solving a cubic with its Hessian and discarding the solution giving the double point tangents, we obtain the following equation of the line of inflexions of the cubic:

$$z = a_3 + (b_5 + 2\mathfrak{C}'a_3)x + (b_6 + 2\mathfrak{B}'a_3)y = 0, \tag{7}$$

where

$$\begin{aligned} b_5 = \frac{-a_3}{8(N\mathfrak{C}'^2 + L\mathfrak{B}'^2)} \{ & -L\mathfrak{B}'[(H-\mathfrak{H}) - (K-\mathfrak{K})] \\ & + 2\mathfrak{C}'[12(N\mathfrak{C}'^2 + L\mathfrak{B}'^2) + N\mathfrak{C}'(\log \mathfrak{C}'r^{1/2})_u \\ & + L\mathfrak{B}'(\log \mathfrak{B}'r^{-1/2})_v]\}, \end{aligned} \tag{8}$$

$$\begin{aligned} b_6 = \frac{-a_3}{8(N\mathfrak{C}'^2 + L\mathfrak{B}'^2)} \{ & N\mathfrak{C}'[(H-\mathfrak{H}) - (K-\mathfrak{K})] \\ & + 2\mathfrak{B}'[12(N\mathfrak{C}'^2 + L\mathfrak{B}'^2) + N\mathfrak{C}'(\log \mathfrak{C}'r^{1/2})_u \\ & + L\mathfrak{B}'(\log \mathfrak{B}'r^{-1/2})_v]\}. \end{aligned}$$

Any line of the second canonical pencil[6] of Davis may have its equation in the form

$$z = 1 + k(\mathfrak{C}'x + \mathfrak{B}'y) = 0,$$

where k is a constant. The flex-ray (7) is a line of this pencil in case

$$2\mathfrak{C}' + b_5/a_3 = k\mathfrak{C}', \qquad 2\mathfrak{B}' + b_6/a_3 = k\mathfrak{B}'.$$

Eliminating k from these equations, we obtain

$$(\mathfrak{B}'^2 + r\mathfrak{C}'^2)[(H-\mathfrak{H}) - (K-\mathfrak{K})] = 0.$$

Therefore, *necessary and sufficient conditions for the second edge of Green to be a line of the second canonical pencil of Davis are*

$$\begin{aligned} & H - \mathfrak{H} = K - \mathfrak{K}, \\ & [r\mathfrak{C}'(\log \mathfrak{C}'r^{1/2})_u + \mathfrak{B}'(\log \mathfrak{B}'r^{-1/2})_v]/(\mathfrak{B}'^2 + r\mathfrak{C}'^2) = \text{const.} \end{aligned} \tag{9}$$

A line of the first canonical pencil[7] of Davis joins the points $(1, 0, 0, 0)$ and $(0, -k\mathfrak{C}'/L, -k\mathfrak{B}'/N, 1)$. It can be verified that the

[6] W. M. Davis, *Contributions to the Theory of Conjugate Nets*, Chicago Doctor's Thesis, 1933, p. 18.

[7] Davis, ibid., p. 20.

first edge of Green is the line joining the point (1, 0, 0, 0) to the point $[0, -(b_6/a_3+4\mathfrak{C}')/L, -(b_6/a_3+4\mathfrak{B}')/N, 1]$. The first edge of Green is a line of the first canonical pencil of Davis if conditions (9) hold. This is a special case of the following theorem.[8]

If one line of an arbitrarily selected pair of lines which are reciprocal with respect to a quadric of Darboux is a line of the second canonical pencil of Davis, its reciprocal is a line of the first canonical pencil of Davis, and conversely.

4. **The associate line of collineation.** The cubic surface

$$z = \tfrac{1}{2}(Lx^2 + Ny^2) + \tfrac{4}{3}(L\mathfrak{C}'x^3 + N\mathfrak{B}'y^3)$$

obtained by truncating the power series (3) can be interpreted[9] geometrically as follows: first, it has third order contact with the analytic surface S given by equations (3) at the point (1, 0, 0, 0); second, it has a unode of second order at the point (0, 0, 0, 1). This cubic surface intersects the tangent plane $z=0$ in the nodal cubic

$$z = 3(x^2 + ry^2) + 8(\mathfrak{C}'x^3 + r\mathfrak{B}'y^3) = 0$$

which has for its line of inflexion $z= 1+2(\mathfrak{C}'x+\mathfrak{B}'y)=0$. This is the *associate line of collineation*,[10] and is a line of the second canonical pencil of Davis.

5. **A generalization of the canonical quadric of Davis.** Davis has shown[11] that at each point P_x of a conjugate net on an analytic surface S there exists a unique quadric surface which has the following properties: the quadric has second order contact with the surface sustaining the net; the axis and ray of the net at P_x are reciprocal polars with respect to the quadric; and the quadric passes through the point P_y which is the harmonic conjugate of P_x with respect to the focal points of the axis. We shall generalize this definition by showing that at each point P_x of a conjugate net there exists a two-parameter family of quadric surfaces having the following properties: *the quadrics have second order contact with the surface sustaining the net and any line of the second canonical pencil of Davis has as its polar reciprocal with respect to any quadric of this family the line of the first canonical pencil defined by the same value of k.*

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[8] We are indebted to P. O. Bell for this theorem in its general form.

[9] Lane, *Projective Differential Geometry of Curves and Surfaces*, The University of Chicago Press, 1932, p. 296.

[10] Davis, ibid., p. 25.

[11] Davis, ibid., p. 11.

The equation[12] of all quadric surfaces having second order contact with the analytic surface S, given by equation (3), is

$$Lx^2 + Ny^2 + z(-2 + k_2x + k_3y + k_4z) = 0, \tag{10}$$

where k_2, k_3, k_4 are arbitrary constants when P_x is fixed, and functions of u, v when P_x varies over S.

If we demand that the polar reciprocals of any line of the first canonical pencil of Davis with respect to a quadric given by equation (10) be the line of the second canonical pencil of David defined by the same value of k, we obtain $k_2 = -4k\mathfrak{C}'$, $k_3 = -4k\mathfrak{B}'$. Substituting these values in (10), we have

$$Lx^2 + Ny^2 + z(-2 - 4k\mathfrak{C}'x - 4k\mathfrak{B}'y + k_4z) = 0. \tag{11}$$

If $k = 0$, and the quadric passes through the point P_y we have the canonical quadric of Davis; also if $k = -1$, we have the quadrics of Darboux.

It is interesting to note that for $k = -1$ the line in the second canonical pencil of Davis is the line of inflexions of the ray-point cubic.[13]

The second canonical tangent[14] of Davis is the line joining the canonical point $(0, \mathfrak{B}', -\mathfrak{C}', 0)$ to the point $P(1, 0, 0, 0)$. The coordinates of any point on this line are

$$(\lambda, \mu\mathfrak{B}', -\mu\mathfrak{C}', 0), \tag{12}$$

where λ and μ are arbitrary constants. The polar plane of this point with respect to the quadrics whose equation is given by (11) is

$$\mu(L\mathfrak{B}'x - N\mathfrak{C}'y) - \lambda z = 0, \tag{13}$$

which is independent of k.

For $\lambda = 0$, the point given by (12) is the canonical point of Davis and the plane (13) is the canonical plane of Davis.

Consequently, we have the theorem: *the polar plane of a point on the second canonical tangent of Davis with respect to a quadric of the family defined by equation* (11) *is a member of a pencil whose axis is the first canonical tangent of Davis. This plane is dependent on the selection of the point but is independent of the selection of the quadric of the family* (11).

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[12] Davis, ibid., p. 11.

[13] Lane, *Conjugate nets and the lines of curvature*, American Journal of Mathematics, vol. 53 (1931), p. 577.

[14] Davis, ibid., pp. 17–18.

A NOTE ON A THEOREM BY WITT[1]

ROBERT M. THRALL

1. **Introduction.** Let F denote the free group with n generators and let F^c be the cth member of the lower central series[2] of F. Witt[3] has shown that $Q^c = F^c/F^{c+1}$ is a free abelian group with $\psi_c(n) = (1/c)\sum\mu(c/d)n^d$ generators (the summation is over all divisors d of c and μ is the Möbius μ-function).

The set of kth powers in F generates a normal subgroup H_k. Let $F_k = F/H_k$ and $G_{k,c} = F_k/F_k^{c+1}$. We shall call F_k the *free k-group* and $G_{k,c}$ the *free k-group of class c*. It is a consequence of Witt's result that F_k^c/F_k^{c+1}, the central of $G_{k,c}$, is abelian and has at most $\psi_c(n)$ generators. In this note we show that if p is a prime greater than c, and $q = p^a$, then the central of $G_{q,c}$ is of order q^N where $N = \psi_c(n)$. If the prime divisors of k are all greater than c, an analogous result holds for the central of $G_{k,c}$ as a consequence of Burnside's theorem that a nilpotent group is the direct product of its Sylow subgroups.

Let M_c denote the space of tensors of rank c over the $GF[p]$. A homomorphic mapping of M_c upon the central of $G_{p,c}$ is set up and enables one to apply the theory of decompositions of tensor space under the full linear group mod p, to determine all characteristic subgroups of $G_{p,c}$ which lie in its central. This theory is applied to determine all the characteristic subgroups of $G_{p,c}$ for $c < 5$ and a multiplication table is constructed for $G_{p,3}$.

2. **Commutator calculus.**[4] Let $s_1, s_2, \cdots$ be operators in any group G and set $s_{12} = (s_1, s_2) = s_1^{-1}s_2^{-1}s_1s_2$ and $s_{12\ldots k} = (s_{12\ldots k-1}, s_k)$. $s_{12\ldots k}$ is called a *simple commutator* of *weight k* in the components $s_1, \cdots, s_k$. The group G^k generated by the simple commutators of weight k for all choices of $s_1, \cdots, s_k$ in G is called the kth member of the *lower central series* of G. If $s \in G^k$ but $s \notin G^{k+1}$, then s is said to have *weight k* in G.

For all s_1, s_2, s_3 in G we have

$$(1) \qquad (s_1s_2, s_3) = s_{13}s_{132}s_{23}, \qquad (s_1, s_2s_3) = s_{13}s_{12}s_{123}.$$

Let the weight of s_i be α_i and set $\alpha = \alpha_1 + \cdots + \alpha_k + 1$. The following relations are then true:

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[1] Presented to the Society, April 13, 1940.

[2] For definition see §2 below or [4, p. 49].

[3] [7, p. 153].

[4] The relations in this section are either taken directly from Hall, Magnus, or Witt or are immediate consequences of their theorems. See [4, 6 and 7].

$$(2) \qquad s_{123\cdots k}s_{213\cdots k} \equiv I \bmod G^{\alpha},$$

$$(3) \qquad s_{123\cdots k}s_{231\cdots k}s_{312\cdots k} \equiv I \bmod G^{\alpha},$$

$$(4) \qquad (s_1^{a_1}, s_2^{a_2}, \cdots, s_k^{a_k}) \equiv (s_{12\cdots k})^{a_1a_2\cdots a_k} \bmod G^{\alpha}.$$

If now $\alpha-1=km$, $m=$minimum $(\alpha_1, \cdots, \alpha_k)$ and $\rho_\beta=\prod_{\delta=1}^{\delta=n}s_\delta^{a_{\delta\beta}}$, $\beta=1, \cdots, k$, it follows that

$$(5) \qquad \rho_{12\cdots k} \equiv \prod_{\beta_\nu=1}^{n} (s_{\beta_1\cdots\beta_k})^{a_{1\beta_1}\cdots a_{k\beta_k}} \bmod G^{\alpha}.$$

3. **The groups F_q.** Let F be the free group generated by $s_1, \cdots, s_n$, and denote by $\overline{H}_k$ the smallest normal subgroup containing the kth powers of all simple commutators of $s_1, \cdots, s_n$.

LEMMA I. *Let $q=p^\alpha$, p any prime. Then $s^q\in\overline{H}_q\cup F^p$ for any element $s\in F$.*

PROOF BY INDUCTION. The lemma is trivial for s of weight greater than $p-1$. Suppose the lemma true for all weight greater than c and let s be of weight c. By the definition of weight, s can be written in the form $s=t_1\cdots t_mv_0$ where v_0 has weight greater than c and the t_i are of weight c and are all simple commutators in $s_1, \cdots, s_n$. Then by the fundamental expansion formula[5] for $(PQ\cdots)^x$ we have

$$s^q = t_1^q\cdots t_m^q v_0^q v_1^q\cdots v_j^q w$$

where $w\in F^p$ and the v_β are all of weight greater than c. By definition $t_\beta^q\in\overline{H}_q$ and by our induction hypothesis $v_\beta^q\in\overline{H}_q\cup F^p$ and so $s^q\in\overline{H}_q\cup F^p$.

COROLLARY I. *Let s have weight c, for $c<p$. Then $s^q\in\overline{H}_q\cup F^{c+1}$.*

Set $H_{q,c}=H_q\cap F^c$ and $\overline{H}_{q,c}=\overline{H}_q\cap F^c$. Then we have

COROLLARY II. *For $c<p$, $H_{q,c}\cup F^{c+1}=\overline{H}_{q,c}\cup F^{c+1}$.*

LEMMA II. $F_q^c/F_q^{c+1}\simeq F^c/(F^{c+1}\cup H_{q,c})$.

We note first that applying the second homomorphism theorem[6] to Hall's formula[7] $F_q^c=(F^c\cup H_q)/H_q$ we obtain the result $F_q^c=F^c/H_{q,c}$ (for all c). Now

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[5] See [4, formula 3.51] or [6, p. 111].

[6] See [2, p. 32].

[7] See [4, formula 2.491] or [2, p. 119].

$$F^c/(F^{c+1} \cup H_{q,c}) \simeq (F^c/H_{q,c})/([F^{c+1} \cup H_{q,c}]/H_{q,c})$$
$$\simeq F_q^c/(F^{c+1}/[H_{q,c} \cap F^{c+1}]) = F_q^c/F_q^{c+1},$$

since $H_{q,c} \cap F^{c+1} = H_{q,c+1}$.

Set $Q_q^c = F_q^c/F_q^{c+1}$.

THEOREM I. *For $c<p$, Q_q^c is abelian of order q^N, $N=\psi_c(n)$.*

DEFINITION. *$t_1, \cdots, t_k$ is said to be a basis for F^c* mod *F^{c+1}, if any operator t of weight c can be written uniquely in the form $t=\prod t_i^{d_i}\theta$ where $\theta \in F^{c+1}$.*

Evidently such a basis exists, and by Witt's theorem[8] $k=N$; and we may choose the t_i as simple commutators in the generators $s_1, \cdots, s_n$. Let ρ_i be the image in Q_q^c of t_i. Then since the t_i are a basis for F^c mod F^{c+1}, any operator $\rho \in Q$ can be written in the form $\rho=\prod \rho_i^{d_i}$ where $0 \leqq d_i < q$. Hence the order of Q_q^c is at most q^N for any c. If the order of Q_q^c is less than q^N there exists a relation $\prod \rho_i^{d_i}=I$ where, say, $d_j \not\equiv 0$.

If now $p>c$, this relation together with Corollary II and Lemma II imply that $\prod t_i^{d_i} \in H_{q,c} \cup F^{c+1}$, or $\prod t_i^{d_i} \equiv \prod t_i^{qe_i}$ mod F^{c+1}. Since the t_i are a basis for F^c mod F^{c+1} this requires $d_i - qe_i = 0$, $i=1, \cdots, N$, which contradicts the assumption that d_j and, therefore, $d_j - qe_j$ is not divisible by q. Hence there can be no relation between the ρ_i and the theorem is proved.

COROLLARY III. *For $p>c$, $G_{q,c}^J$ is of order q^m,*

$$m = \psi_j(n) + \cdots + \psi_c(n), \qquad j = 1, \cdots, c.$$

4. **Characteristic subgroups of $G=G_{p,c}$.** A large variety of characteristic subgroups of G can be obtained from the lower central series by sequences of joins, intersections, and commutations. In G the upper and lower central series are identical; in particular, the central C $(=C_{p,c})$ of G is G^p. The central quotient group of G is $G_{p,c-1}$, and any characteristic subgroup H of G is mapped into a characteristic subgroup $H'=H \cup C/C$ in $G_{p,c-1}$.

We say that K is a *minimal characteristic subgroup* (m.c.s.) of G if no proper subgroup of K is characteristic in G. *For $G=G_{p,c}$, every m.c.s. lies in the central.* Indeed any normal subgroup of G must contain commutators of weight c and therefore must have an intersection not equal to I with C. We turn now to the determination of all characteristic subgroups of G which lie in C.

[8] See [7, Theorems 3 and 4, pp. 152–153].

Let $\overline{A}$ be any automorphism of G, and H any characteristic subgroup of G. $\overline{A}$ induces an automorphism $\overline{A}(H)$ on G/H and an automorphism $\overline{A}[H]$ on H. If in particular H is G^2, the commutator subgroup of G, then G/H is the abelian group of order p^n and type $1, 1, 1, \cdots$. Let the generators of G be $s_i, \cdots, s_n$, and let t_i be the image in G/G^2 of s_i. Then $\overline{A}(H)$ takes the form $t_i \to t_i'$ where

$$t_i' = \prod t_j^{a_{ij}}, \quad a_{ij} \in GF[p], \qquad |a_{ij}| \neq 0.$$

Hence $\overline{A}$ itself must be of the form $s_i \to s_i'$ where

$$s_i' = \prod s_j^{a_{ij}} r_i, \qquad r_i \in G^2.$$

To calculate $\overline{A}[C]$ we apply (5) with $k = c$. Since $G^{c+1} = I$, (5) is now an equality and shows that $\overline{A}[C]$ is independent of the r_i. Indeed if we set $A = (a_{ij})$ we see that the formal commutators $s_{i_1 \cdots i_c}$ transform like tensors of rank c, that is, according to $A \times A \times \cdots \times A$ (Kronecker direct product with c factors).

Denote by M_c the whole space of tensors of rank c. It has dimension n^c. The group $A_c = \{A \times A \times \cdots \times A\}$ (c factors) is homomorphic to the group $\{A\}$ of linear transformations, and hence M_c is a representation space for $\{A\}$. Brauer[9] has proved the following theorem concerning the decompositions of this representation:

THEOREM II. *If K is a field of characteristic $p \neq 0$, the representation A_c is completely reducible for $c < p$, and it splits into irreducible parts in exactly the same way as in the case of characteristic zero.*

The mapping $x_{i_1 \cdots i_c} \to s_{i_1 \cdots i_c}$ (where of course products in C are replaced by sums in M_c) establishes a homomorphic mapping of M_c upon C and this mapping is preserved under the group A_c, that is, C is also a representation space for the group A_c. Let $\overline{C}$ denote C written additively. Then $\overline{C} = M_c - W_c$, where W_c contains all tensors whose image in C is identity. We call W_c the *space of commutator relations*, W_c is evidently an invariant subspace of M_c under the tensor group and by Theorem I it has dimension $n^c - \psi_c(n)$ if $p > c$. Because of the complete reducibility of the representation A_c we can write $M_c = W_c + P_c$ where P_c is likewise an invariant subspace of M_c, and furthermore the decomposition into irreducibly invariant subspaces of P_c under A_c will be the same as that of C under the group of automorphisms of G. (P_c is not uniquely determined by W_c but its decompositions are.) Let $R_1, \cdots, R_t$ be irreducibly invariant sub-

[9] See [3, p. 867].

spaces of M_c whose direct sum is P_c, and let $T_1, \cdots, T_t$ be the corresponding subgroups of C. Then the following theorem expresses the above arguments in group theoretic terms:

THEOREM III. *Any minimal characteristic subgroup is isomorphic to one of $T_1, \cdots, T_t$ and any characteristic subgroup K of G which lies in the central is the direct product of the minimal characteristic subgroups which it contains. ($p>c$ is assumed throughout.)*

The number of characteristic subgroups in G is clearly independent of the number n of generators provided that $n \geqq c$. Hence to obtain all characteristic subgroups of the set of groups $G_{p,c}$ with $p>c$ we need only consider those with $n=c$.

5. **The groups $G_{p,3}$ and $G_{p,4}$.** In this section we shall make use of the decomposition into irreducibly invariant subspaces of the tensor spaces M_3 and M_4. These can be readily obtained by a direct computation based upon the decomposition theorems of M_c in general.[10] We suppose $n=3$ in M_3 and $n=4$ in M_4.

$M_3=\sum_1+\sum_{2,1}+\sum_{2,2}+\sum_3$ in which the summands have dimensions 10, 8, 8 and 1 respectively. $W_3=\sum_1+\sum_{2,1}+\sum_3$ and hence $G_{p,3}$ has just one m.c.s., its central.

$$M_4 = \sum_1 + \sum_{2,1} + \sum_{2,2} + \sum_{2,3} + \sum_{3,1} + \sum_{3,2} + \sum_{4,1} + \sum_{4,2} + \sum_{4,3} + \sum_5$$

in which the summands have dimensions 35, 45, 45, 45, 20, 20, 15, 15, 15, and 1 respectively. $W_4=\sum_1+\sum_{2,1}+\sum_{2,2}+\sum_{3,1}+\sum_{3,2}+\sum_{4,1}+\sum_{4,2}+\sum_5$ and hence $G_{p,4}$ has two m.c.s., one of which is its second derived group. Let us denote these by D and E.

$G_{p,1}$ has no proper characteristic subgroups and the only proper characteristic subgroup of $G_{p,2}$ is its central $G^2_{p,2}$.

THEOREM IV. *The only characteristic subgroups of $G_{p,3}$ are the members of its lower central series.*

Let H be characteristic in $G_{p,3}$. Then if $H \neq I$ or C, by Theorem III $H \supset C$. $H'=H/C$ must then be $G_{p,2}$ or its central. In the first case $H=G_{p,3}$ and in the second case $H=G^2_{p,3}$.

THEOREM V. *The only characteristic subgroups of $G_{p,4}$ are D, E and the members of the lower central series.*

It is easy to see that if a characteristic subgroup $H \supset C$ then H is in

[10] See for instance [1, Theorem 4.4D, p. 129].

the lower central series. To complete the proof we show then that if $H \not\supset C$, $H = D$ or E. Since $H \not\supset C$, either $H' = I$; in which case $H \subset C$ and therefore $H = D$ or E; or $H' \supset G^3_{p,3}$ (by Theorem IV). It remains now only to show that $H' \supset G^3_{p,3}$ implies $H \supset C$. If now $H' \supset G^3_{p,3}$, then $H \cup C \supset G^3_{p,4}$ and hence for the commutator s_{122} of weight 3 we have a factorization $s_{123} = hd$ where $h \in H$ and $d \in C$ (and so d has weight not less than 4). Since H is normal $(h, s_4) = (s_{123} \cdot d^{-1}, s_4) = s_{1234} \in H$. But the conjugates of s_{1234} generate C so that $H \supset C$ contrary to hypothesis, and the theorem is proved.

For the sake of completeness we give a multiplication table for $G_{p,3}$. Applying the formulas of §2 and Theorem I we have for any operator s of $G_{p,3}$ a unique expression in the form

$$s = s^{A} = \prod s_i^{a_i} \prod_{i<j} s_{ij}^{a_{ij}} \prod_{i \neq j} s_{iij}^{a_{iij}} \prod_{i<j<k} s_{ijk}^{a_{ijk}} s_{jki}^{a_{jki}}.$$

If now $s^C = s^A s^B$, then applying the readily verified formula $(s_1^\alpha, s_2^\beta) = s_{12}^{\alpha\beta} s_{121}^{\binom{\alpha}{2}\beta} s_{122}^{\alpha\binom{\beta}{2}}$ we obtain[11] $(i<j<k)$

$$\begin{aligned}
c_i &= a_i + b_i, \qquad c_{ij} = a_{ij} + b_{ij} - b_i a_j, \\
c_{iij} &= a_{iij} + b_{iij} - b_i C_{a_j,2} + b_j a_{ij} - b_i b_j a_j, \\
c_{jii} &= a_{jii} + b_{jii} + a_j C_{b_j,2} - b_i a_{ij}, \\
c_{ijk} &= a_{ijk} + b_{ijk} + b_j a_{ik} + b_k a_{ij} - b_i a_j a_k - b_i b_j a_k - b_i b_k a_j, \\
c_{jki} &= a_{jki} + b_{jki} + b_i a_{jk} + b_j a_{ik} - b_i b_j a_k.
\end{aligned} \tag{6}$$

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[11] For $p = 3$, $s_{ijj} = s_{jii} = I$ and $s_{ijk} = s_{jki}$ so that (6) reduces to formula 9 of Levi and van der Waerden [5, p. 156].

NON-INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH A $Q_{1,2}$ CONGRUENCE

A. B. CUNNINGHAM

De Paolis[1] discussed the involutorial transformations associated with the congruence of lines meeting a curve of order m and an $(m-1)$-fold secant, while Vogt[2] studied the transformation T for a linear congruence and bundle of lines. In the present paper the transformations associated with the congruence of lines on a conic and a secant of it are discussed.

Given a conic r, a line s meeting r once, and two projective pencils of surfaces

$$|F_{n+m+1}| : r^n s^m g; \qquad |F'_{n'+m'+1}| : r^{n'} s^{m'} g',$$

where $n \leqq m+1$, $n' \leqq m'+1$, $[r, s]=A$, and g, g' the residual base curves.

Through a generic point P, there passes a single surface F of $|F|$. The unique line t through P, r, s meets the associated F' in one residual point P', image (T) of P. The transformations to be considered are of three types:

Case I. $n=m+1$, $n'=m'+1$.
Case II. $n<m+1$, $n'<m'+1$.
Case III. $n=m+1$, $n'<m'+1$.

Case I

Given

$$|F_{2n}| : r^n s^{n-1} g; \qquad |F'_{2n'}| : r^{n'} s^{n'-1} g';$$

where g, g' are of order n^2+2n-1, $n'^2+2n'-1$. The curve g meets r, s in n^2+2n-1, n^2-1 points respectively.

The conic r is a fundamental curve whose image (T^{-1}) is $R: r^{n+n'}$, since there are $(n+n')$ invariant directions through each point on r. R is generated by a monoidal plane curve of order $n+n'+1$, one curve on each plane of the pencil $(O,s)=w$, as O, describes r. The fundamental line s has for image (T^{-1}) a surface $S: s^{n+n'-1}$, of which $n+n'-2$ branches are invariant. A is a fundamental point of the first kind, whose image (T^{-1}) is the plane $u: r$. In the plane $v: s$ and tangent

[1] De Paolis, *Alcuni particolari trasformazioni involutori dello spazio*, Rendiconti dell' Accademia dei Lincei, Rome, (4), vol. 1 (1885), pp. 735–742, 754–758.

[2] Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterländische Kultur, class 84, 1906, pp. 8–16.

to r there is a curve $C_{n+n'}$, image (T^{-1}) of the intersection of r, s at A, which lies on R, S. The tangent line $[u, v]$ to r at A lies on the surface R.

From any point Q' on g', there is a unique transversal t meeting r, s. Any point P on t determines an F and t meets the associated F' in a residual point Q', thus $Q' \sim (T^{-1})t$. Every point P' on t determines the same F' and t meets the associated F in one point $\overline{P}$; thus $\overline{P} \sim (T)t$. Considering all points on g'

$$g' \sim (T^{-1})G; \qquad \bar{g}_x \sim (T)G,$$

where $\bar{g}_x$ is the locus of points $\overline{P}$. Similarly

$$g \sim (T)G'; \qquad \bar{g}'_y \sim (T^{-1})G'.$$

The eliminant of the parameter from $|F|$, $|F'|$ is a point-wise invariant surface $K_{2n+2n'}$. A generic plane meets every line of the pencil (Au), hence the homaloidal surfaces have an additional fixed direction d through A.

The table of characteristics for T^{-1} is

$\pi' \sim$	$\phi_{2n+2n'+2}$:	$A^{n+n'+1+d}$	$r^{n+n'+1}$	$s^{n+n'}$	g	$\bar{g}_x$,		
$K \sim$	$K_{2n+2n'}$:	$A^{n+n'}$	$r^{n+n'}$	$s^{n+n'-2}$	g	$\bar{g}_x$	g'	$\bar{g}'_y$,
$r \sim$	$R_{2n+2n'+1}$:	$A^{n+n'+d}$	$r^{n+n'}$	$s^{n+n'}$	g	$\bar{g}_x$	$C_{n+n'}$	$[u, v]$,
$s \sim$	$S_{2n+2n'}$:	$A^{n+n'}$	$r^{n+n'}$	$s^{n+n'-1}$	g	$\bar{g}_x$	C,	
$g' \sim$	$G_{4n'}$:	$A^{2n'}$	$r^{2n'}$	$s^{2n'}$	g'	$\bar{g}_x$,		
$\bar{g}'_y \sim$	G_{4n}:	A^{2n}	r^{2n}	s^{2n}	g	$\bar{g}'_y$,		
$A \sim$	u:	A	r,					

$J \equiv u^3RSGG'$.

The intersection of two ϕ'-surfaces gives the order of $\bar{g}'_y$, $y = n^2 + 2nn' + 2n + 1$. The curve $\bar{g}'_y$ meets r, s in y, $y - 2n$ points respectively.

The equations of T^{-1} are $\tau x_i = Ry_i - Kz_i = Suy_i + Kw_i$, where z_i, w_i are the points $[t, r]$, $[t, s]$.

Case II

Given

$$|F_{n+m+1}| : \quad r^n s^m g; \qquad |F'_{n'+m'+1}| : \quad r^{n'} s^{m'} g',$$

where g, g' are of order $2mn + 2m + 2n - n^2 + 1$, $2m'n' + 2m' + 2n' - n'^2 + 1$. The curve g meets r, s in $2mn + 4n - n^2$, $2mn + 2m - n^2$ points respectively.

A is a fundamental point of the second kind with image $(T^{-1})C_{n+n'+1}$: $A^{n+n'}$ in the plane v.

The image (T^{-1}) of a point on s is a curve $s_{m+m'+2}$ on the quadric cone on r, with a $(m+m')$-fold point at the vertex and one point on each generator. This curve generates the surface S. The equations of T are

$$\tau x = Ry_i - Kz_i = Sy_i + Kw_i.$$

The table of characteristics for T^{-1} is

$$\begin{array}{lrlllll}
\pi' \sim \phi_{n+n'+m+m'+4}: & r^{n+n'+1} & s^{m+m'+2} & g & \bar{g}_x, & & \\
K \sim K_{n+n'+m+m'+2}: & r^{n+n'} & s^{m+m'} & g & \bar{g}_x & g' & \bar{g}'_y, \\
r \sim R_{n+n'+m+m'+3}: & r^{n+n'} & s^{m+m'+2} & g & \bar{g}_x & C_{n+n'+1}, & \\
s \sim S_{n+n'+m+m'+3}: & r^{n+n'+1} & s^{m+m'+1} & g & \bar{g}_x & C_{n+n'+1}, & \\
g' \sim G_{2n'+2m'+3}: & r^{2n'+1} & s^{2m'+2} & g' & \bar{g}_x, & & \\
\bar{g}'_y \sim G'_{2n+2m+3}: & r^{2n+1} & s^{2m+2} & g & \bar{g}'_y, & &
\end{array}$$

$$J \equiv RSGG',$$

where $y = 2mn+2m'n+2mn'+3m+3n+m'+n'-n+5-2nn'$. The curve $\bar{g}'_y$ meets r, s in $[y-(2m-2n+1)]$, $[y-(2n+1)]$ points respectively.

Case III

Given

$$|F_{2n}|: \; r^n s^{n-1} g; \qquad |F'_{n'+m'+1}|: \; r^{n'} s^{m'} g',$$

where g, g' are of order n^2+2n-1, $2m'n'+2m'+2n'-n'^2+1$. The curve g meets r, s in n^2+2n-1, n^2-1 points, and g' meets r, s in $2m'n'+4n'-n'^2$, $2m'n'+2m'-n'^2$ points respectively.

In T^{-1} (T) A is a fundamental point of the second (first) kind with image $C'_{n+n'}$ (u). For some point D on a line $\overline{P'A}$ of the pencil (Au), the associated F is the one determined by the direction $\overline{P'A}$; thus $D \sim (T^{-1})\overline{P'A}$. The locus of D is a curve $\delta_{m'-n'+1}$: $A^{m'-n'}$ such that $\delta \sim (T^{-1})u$.

Since $[r, \delta] = (m'-n'+2)$ points aside from A, R: $(m'-n'+2)$ lines of the pencil (Au), hence R: $A^{n+m'+2}$. The image (T^{-1}) of A as a point on s is $C_{n+n'+1}$ and the $(m'-n')$ tangents to δ at A, hence S: $A^{n+m'+1}$.

For the $(2m'-2n'+1)$ points, aside from those on r, in which g' meets u, t becomes a line of the pencil (Au). Therefore $\bar{g}_x$: $A^{2m'-2n'+1}$ and $[g', \delta] = (2m'-2n'+1)$ points.

The table of characteristics for T^{-1} is

$$\begin{array}{lrllllll}
\pi' \sim \phi_{2n+n'+m'+3}: & A^{n+m'+1} & r^{n+n'+1} & s^{n+m'+1} & g & \bar{g}_x, & & \\
K \sim K_{2n+n'+m'+1}: & A^{n+m'} & r^{n+n'} & s^{n+m'-1} & g & \bar{g}_x & g' & \bar{g}_y'\delta, \\
r \sim R_{2n+n'+m'+2}: & A^{n+m'+2} & r^{n+n'} & s^{n+m'+1} & g & \bar{g}_x & C_{n+n'+1}, & \\
s \sim S_{2n+n'+m'+2}: & A^{n+m'+1} & r^{n+n'+1} & s^{n+m'} & g & \bar{g}_x & C_{n+n'+1}, & \\
g' \sim G_{2n'+2m'+3}: & A^{2m'+2} & r^{2n'+1} & s^{2m'+2} & g' & \bar{g}_x, & & \\
\bar{g}_y' \sim G_{4n}': & A^{2n} & r^{2n} & s^{2n} & g & \bar{g}_y', & & \\
\delta \sim u: & A & r & \delta, & & & &
\end{array}$$

$$J \equiv uRSGG',$$

where $y=n^2+2m'n+4n+1$. The curve $\bar{g}_y'$ meets r, s in y, $y-2n$ points respectively. The equations of T^{-1} are $\tau x=Ry_i-Kz_i=Sy_i+Kw_i$.

The table of characteristics for T is

$$\begin{array}{llllll}
\pi \sim \phi_{2n+n'+m'+3}': & r^{n+n'+1} & s^{n+m'+1} & g' & \bar{g}_y' & \delta, \\
r \sim R_{2n+n'+m'+2}: & r^{n+n'} & s^{n+m'+1} & g' & \bar{g}_y' & C_{n+n'}' \quad [u, v]\delta, \\
s \sim S_{2n+n'+m'+1}': & r^{n+n'} & s^{n+m'} & g' & \bar{g}_y' & C_{n+n'}', \\
g \sim G_{4n}', & \bar{g}_x \sim G_{2n'+2m'+3}, & & & & \\
A \sim u: \quad Ar\delta, & J' \equiv u^2R'S'G'G, & & & &
\end{array}$$

where $x=2m'n'+2m'n-n'^2+3m'+n'+2n+4$. The curve $\bar{g}_x$ meets r, s in $x-(2m'-2n'+1)$, $x-(2m+2)$ points respectively. The equations of T are $\tau' y=R'x_i+Kz_i' =S'ux_i-Kw_i'$.

In each of the three cases there exists a monoidal transformation in the plane w. The space transformations are generated by allowing the vertex to describe the conic r, and the plane to generate the pencil on s.

WEST VIRGINIA UNIVERSITY

REFLEXIVE BANACH SPACES NOT ISOMORPHIC TO UNIFORMLY CONVEX SPACES[1]

MAHLON M. DAY[2]

Clarkson[3] introduced the notion of uniform convexity of a Banach space: B is uniformly convex if for each ϵ with $0<\epsilon\leqq 2$ there is a $\delta(\epsilon)>0$ such that whenever $\|b\|=\|b'\|=1$ and $\|b-b'\|\geqq\epsilon$, then $\|b+b'\|\leqq 2(1-\delta(\epsilon))$. Milman[4] and Pettis[5] have demonstrated that any uniformly convex space is reflexive;[6] that is, that for each $\beta\in B^{**}$ there is a $b\in B$ with $\beta(f)=f(b)$ for every $f\in B^*$. The same result clearly holds if B is not uniformly convex but can be given a new norm defining the same topology under which the space is uniformly convex. It has been conjectured that every reflexive space can be given such a topologically equivalent uniformly convex norm; that is, that, in Banach's terminology,[7] every reflexive space is isomorphic to a uniformly convex space. We shall show by a large class of examples that this is not the case; in fact the following result holds:

THEOREM 1. *There exist Banach spaces which are separable, reflexive, and strictly convex,*[8] *but are not isomorphic to any uniformly convex space.*

We shall start with a class of Banach spaces and pick out a simple example having all but the strict convexity property; with this as a sample of what can happen we easily find a large number of spaces satisfying all the conditions of the theorem. As an application of our results we show that certain ergodic theorems of Alaoglu and Birkhoff[9] can be extended to some non-uniformly convex spaces.

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[1] Presented to the Society, November 23, 1940, under the title *Some more uniformly convex spaces.*

[2] Corinna Borden Keen Research Fellow of Brown University.

[3] J. A. Clarkson, *Uniformly convex spaces*, Transactions of this Society, vol. 40 (1936), pp. 396–414.

[4] D. Milman, *On some criteria for the regularity of spaces of the type* (B), Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, new series, vol. 20 (1938), pp. 243–246.

[5] B. J. Pettis, *A proof that every uniformly convex space is reflexive*, Duke Mathematical Journal, vol. 5 (1939), pp. 249–253.

[6] That is, "regular" in the terminology of Hahn. We use B^* to denote the conjugate space of the Banach space B; $B^{**}=(B^*)^*$.

[7] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, chap. 11.

[8] B is strictly convex if the set of points with $\|b\|=1$ contains no line segments; that is, if $\|b\|=\|b'\|=1$ and $b''=tb+(1-t)b'$ with $0<t<1$, then $\|b''\|<1$.

[9] L. Alaoglu and G. Birkhoff, *General ergodic theorems*, Annals of Mathematics, (2), vol. 41 (1940), pp. 293–309.

Let $\{B_i\}$, $i=1, 2, \cdots$, be a sequence of Banach spaces and let p be a real number >1. Let $B=\mathcal{P}^p\{B_i\}$ be the class of all those sequences $\boldsymbol{b}=\{b_i\}$ with $b_i\in B_i$ and $\|\boldsymbol{b}\|=(\sum_{i<\infty}\|b_i\|^p)^{1/p}<\infty$. It is easily verified that, with this norm and the obvious definitions of vector operations, B is a Banach space.

LEMMA. *$\mathcal{P}^p\{B_i\}$ is reflexive (strictly convex) if and only if all B_i are reflexive (strictly convex); in fact, $(\mathcal{P}^p\{B_i\})^*=\mathcal{P}^{p'}\{B_i^*\}$, where $1/p+1/p'=1$.*

For reflexivity we need only prove the last statement. If $\boldsymbol{f}$ is linear on $B=\mathcal{P}^p\{B_i\}$, let $\boldsymbol{b}_i$ be any element of B of the form $(0, 0, \cdots, b_i, 0, 0, \cdots)$ and define $f_i\in B_i^*$ by $f_i(b_i)=\boldsymbol{f}(\boldsymbol{b}_i)$; then there exist $b_i\in B_i$ with $\|b_i\|=1$ and $\|f_i\|/2\leqq f_i(b_i)\leqq\|f_i\|$ for all i. Since $\|b_i\|=1$ the sequence $\{\alpha_i b_i\}$ is in $\mathcal{P}^p\{B_i\}$ if and only if the sequence of real numbers $\{\alpha_i\}$ is in l^p. If $\boldsymbol{b}^j=(\alpha_1 b_1, \alpha_2 b_2, \cdots, \alpha_j b_j, 0, 0, \cdots)$ and $\boldsymbol{b}=\{\alpha_i b_i\}\in\mathcal{P}^p\{B_i\}$, then $\|\boldsymbol{b}^j-\boldsymbol{b}\|\to 0$ so $\boldsymbol{f}(\boldsymbol{b}^j)\to\boldsymbol{f}(\boldsymbol{b})$; clearly $\boldsymbol{f}(\boldsymbol{b}^j)=\sum_{i\leqq j}\alpha_i f_i(b_i)$, so the series $\sum_{i<\infty}\alpha_i f_i(b_i)$ converges if $\sum|\alpha_i|^p<\infty$. Hence the sequence $\{f_i(b_i)\}\in l^{p'}$ and therefore the sequence $\{\|f_i\|\}\in l^{p'}$; so to each $\boldsymbol{f}\in[\mathcal{P}^p\{B_i\}]^*$ we have associated a point $\{f_i\}$ of $\mathcal{P}^{p'}\{B_i^*\}$ such that for any $\boldsymbol{b}=\{b_i\}$, $\boldsymbol{f}(\boldsymbol{b})=\sum_i f_i(b_i)$. Conversely if $\{f_i\}\in\mathcal{P}^{p'}\{B_i^*\}$,

$$\sum_{i<\infty}|f_i(b_i)|\leqq\sum_{i<\infty}\|f_i\|\cdot\|b_i\|\leqq\left(\sum_i\|f_i\|^{p'}\right)^{1/p'}\left(\sum_i\|b_i\|^p\right)^{1/p}$$

by Hölder's inequality; so this sequence $\{f_i\}$ defines an $\boldsymbol{f}\in(\mathcal{P}^p\{B_i\})^*$, with $\boldsymbol{f}(\boldsymbol{b})=\sum_i f_i(b_i)$; clearly $\|\boldsymbol{f}\|=(\sum_i\|f_i\|^{p'})^{1/p'}$, so we have completed our proof of reflexivity.

The proof of strict convexity is straightforward if we notice that strict convexity is equivalent to what Krein and Šmulian[10] call "strict normalization"; B is strictly normalized if $\|b+b'\|=\|b\|+\|b'\|$ and $\|b'\|\neq 0$ imply $b=tb'$, $t\geqq 0$.

We now make a simple choice of the spaces B_i to get an example of a reflexive space not isomorphic to a uniformly convex space. Let B_i be the i-dimensional Minkowski space of points $b_i=(b_{i1}, \cdots, b_{ii})$ in which $\|b_i\|=\sup_{j\leqq i}|b_{ij}|$; then each B_i is reflexive since it is finite dimensional, so $B=\mathcal{P}^p\{B_i\}$ is reflexive, by our lemma, for any choice of $1<p<\infty$. This B can not be given a uniformly convex norm defining the same topology.

[10] V. Šmulian, *On some geometrical properties of the unit sphere in the space of the type (B)*, Recueil Mathematique (i.e., Matematicheskii Sbornik), new series, vol. 6 (48) (1940), pp. 90–94.

Suppose that there is such a norm—we shall represent it by $|b|$—defined in B. For $0<\epsilon\leqq 2$ we define $\delta(\epsilon)=\frac{1}{2}\inf(2-|b+b'|)$, where the inf is taken over b, $b'\in B$ with $|b|\leqq 1\geqq|b'|$ and $|b-b'|\geqq\epsilon$; it can easily be shown that $\delta(\epsilon)>0$ for all such ϵ if and only if the norm $|b|$ is uniformly convex. For the topologies to be the same it is clear that there must exist m, M with $0<m\leqq M<\infty$ such that $m\|b\|\leqq|b|\leqq M\|b\|$; without loss of generality we can make $m=1$ by a change of scale of $|b|$ and get $\|b\|\leqq|b|\leqq M\|b\|$ with $1\leqq M<\infty$. If we fix any i this must also hold in B_i since B_i is a closed linear subspace in B; so, if we let K_i be the "sphere" of those points of B_i with $|b_i|\leqq 1$, we see that K_i is contained in the unit cube, $E_{b_i}[\sup_{j\leqq i}|b_{ij}|\leqq 1]$, and contains the cube $E_{b_i}[\sup_{j\leqq i}|b_{ij}|\leqq 1/M]$. If we let $\alpha=1/M$, we know that the points $(\epsilon_1\alpha, \epsilon_2\alpha, \cdots, \epsilon_i\alpha)$, $\epsilon_j=\pm 1$, all lie in K_i. Considering, for example, $b=(\alpha, \cdots, \alpha, \alpha)$ and $b'=(\alpha, \cdots, \alpha, -\alpha)$, we have $b-b'=(0, \cdots, 0, 2\alpha)$, and $(b+b')/2=(\alpha, \cdots, \alpha, 0)$. Now $|b-b'|\geqq\|b-b'\|=2\alpha$, so $\frac{1}{2}|b+b'|\leqq 1-\delta(2\alpha)$; that is, $|(\alpha, \cdots, \alpha, 0)|\leqq 1-\delta(2\alpha)$ or $|(\alpha/(1-\delta(2\alpha)), \cdots, \alpha/(1-\delta(2\alpha)), 0)|\leqq 1$. This could have been done with any choice of ϵ_j, $j\leqq i-1$, so we can repeat $i-2$ more times, noting that $1-\delta(2\alpha)<1$ and that $\delta(\epsilon)$ decreases as ϵ decreases, so that $\delta[\alpha/(1-\delta(2\alpha))^j]\geqq\delta(2\alpha)$, $j=1, \cdots, i-1$; setting $b_i=(\alpha/(1-\delta(2\alpha))^{i-1}, 0, \cdots, 0)$, we have $|b_i|\leqq 1$, but $|b_i|\geqq\|b_i\|>1$ if i is large enough. This contradiction proves our theorem.

Since B is clearly separable if all B_i are, we could apply a theorem of Clarkson to show that an example satisfying all the conditions including strict convexity exists. However we need only take B_i to have an l^{q_i} norm instead of an (m) norm and if $q_i\to\infty$ fast enough ($q_i=i$ will do) then this new example is isomorphic to the old one while, by our lemma, it is strictly convex.

It is clear from the nature of our construction that if we had taken B_i with the l^1 norm instead of the (m) norm a similar contradiction would have appeared if we tried to fit in a uniformly convex norm (just for simplicity in the construction it is easier to take the dimension of B_i to be 2^i in this case). Finally the finite dimensionality of the B_i was used essentially only in proving reflexivity; this and the existence of subspaces of large dimension are our main requirements. Let l^q_μ, μ a cardinal number $\geqq 0$, $1<q<\infty$, be the μ-dimensional space with l^q norm; in this notation the ordinary l^q spaces would be $l^q_{\aleph_0}$. Using our example as a guide we have the following result:

THEOREM 2. *The space $\mathcal{P}^p\{l^{\mu_i}_{q_i}\}$ is not isomorphic to a uniformly convex space, provided there exists a sequence $\{i_j\}$ of indices for which $\lim_j\mu_{i_j}\geqq\aleph_0$ while at the same time either $\lim_j q_{i_j}=+\infty$ or $\lim_j q_{i_j}=1$.*

In the first case we can show that $\mathcal{P}^p\{l^{q_i}_{\mu_i}\}$ has a subspace isomorphic to the example first given while in the second case it contains a subspace isomorphic to the example suggested in the paragraph preceding this theorem. By the same sort of embedding argument we can clearly get the same result for $\mathcal{P}^p\{L^{q_i}\}$ if the q_i are not bounded away from one and infinity.

A recent result of Boas is that $\mathcal{P}^p\{l^{q_i}\}$ is uniformly convex if $q_i=q$ for all i and $1<q<\infty$. It is not yet known whether $\mathcal{P}^p\{l^{q_i}\}$ can be made uniformly convex, or is already so, if the q_i are bounded away from 1 and ∞; i.e., if there exist m, M such that $1<m\leqq q_i\leqq M<\infty$ for all i.

An application to ergodic theory. Milman and Alaoglu and Birkhoff have use for the following lemma:

If K is a closed, convex, non-empty subset of a uniformly convex space, then there is a unique point in K of least norm.

From a theorem of Mazur[11] that any closed, convex set is closed under weak convergence we have the following result.

LEMMA. *If K is a closed convex non-empty set in a Banach space whose unit sphere is weakly compact,*[12] *there is at least one point in K of least norm. If B is also strictly convex, then there is just one such point.*

From this it follows that the general ergodic theorems depending on the minimal methods of proof of Alaoglu and Birkhoff hold in more spaces than the uniformly convex ones; we note the extensions of one or two of their theorems (we number them correspondingly); that the extensions are real and not merely formal follows from our existence theorem.

THEOREM 5. *If G is abelian, if B is strictly convex and has unit sphere weakly compact, and if the norms of the operators T_g are at most one, then every element of B is ergodic.*

THEOREM 6. *If the norms of the operators T_g are at most one, if the unit sphere in B is weakly compact, strictly convex and without sharp edges, then every element of B is ergodic.*

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[11] S. Mazur, *Über konvexe Mengen in linearen normierten Raümen*, Studia Mathematica, vol. 4 (1933), pp. 70–84.

[12] The unit sphere of B is weakly compact if given any sequence $\{b_n\}$ of points with $\|b_n\|\leqq 1$ there is a subsequence $\{b_{n_i}\}$ and a point b_0 (clearly of norm not greater than 1) for which $\lim_i f(b_{n_i})=f(b_0)$ for all $f\in B^*$.

COROLLARY. *If B is any $\mathcal{P}^p\{l^{\mu_i}\}$ or $\mathcal{P}^p\{L^{q_i}\}$, $1<p<\infty$, $1<q_i<\infty$, μ_i any cardinal $\geqq 0$, then every element of B is ergodic.*

Theorem 6 can be stated in a more symmetric form.

THEOREM 6′. *If the norms of the operators T_g are at most one, and if the unit spheres in both B and B^* are weakly compact and strictly convex, then every element of B is ergodic.*

By a theorem of Gantmakher and Šmulian[13] weak compactness of the unit sphere in both B and B^* follows from the weak compactness of either unit sphere; so this can be weakened slightly. Alaoglu and Birkhoff in a footnote remark that the condition that the unit sphere in B have no sharp edges is equivalent to the condition that the unit sphere in B^* be strictly convex. Note that a theorem of Milman asserts that a Banach space with a weakly compact and strictly convex unit sphere is reflexive.

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[13] V. Gantmakher and V. Šmulian, *Sur les espaces linéaires dont la sphère unitaire est faiblement compact*, Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, new series, vol. 17 (1937), pp. 91–94.

DIFFERENTIAL AND INTEGRAL INVARIANTS OF PLANE CURVES AND HORN ANGLES[1]

J. M. FELD

1. Introduction. A horn angle has been defined by E. Kasner as a figure formed by an ordered pair of oriented analytic arcs which pass through a common point in a common direction and which have contact of order $n \geqq 1$ at the common point. The order of contact is said to be the order of the horn angle. Thus, in a horn angle of the first order the arcs forming its sides have, at the vertex of the angle, the same inclination (direction) but different curvatures. Horn angles of the first order have a unique absolute conformal invariant, first found by Kasner.[2] Two relative conformal invariants of horn angles of the first order were given later in a joint paper by Kasner and Comenetz.[3] Kasner has also determined conformal invariants of the fifth order for horn angles of second order contact.[4]

Kasner and Comenetz obtained the above-mentioned invariants by the use of power series and by repeated differentiation of the Cauchy-Riemann equations. It is shown in §2 of this paper that the relative and absolute conformal invariants of horn angles of the first order can be obtained directly from two standard formulas; one of these expresses d^2y/dx^2 in terms of u and v where $x=x(u)$ and $y=y(v)$; the other expresses the Schwarzian derivative

$$\{y, x\} = \frac{d^3y}{dx^3}\Big/\frac{dy}{dx} - \frac{3}{2}\left(\frac{d^2y}{dx^2}\Big/\frac{dy}{dx}\right)^2$$

in terms of u and v under the same conditions. As a consequence, the invariants are presented in a form which permits of their being interpreted as invariants, not only under the conformal group, but also under a group of multiply differentiable transformations of the type: $x=x(u)$, $y=y(v)$, where x is a function of u alone, and y is a function of v alone.[5]

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[1] Presented to the Society, February 25, 1939.

[2] E. Kasner, *Conformal geometry*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912, vol. 2, p. 81.

[3] E. Kasner and G. Comenetz, *Conformal geometry of horn angles*, National Academy of Sciences, vol. 22 (1936), p. 303. See also Comenetz, *Kasner's invariant and trihornometry*, American Mathematical Monthly, vol. 45 (1938), p. 28.

[4] Kasner, Transactions of this Society, vol. 44 (1938), p. 25.

[5] Transformations of this type occur, for example, in the papers: Kasner, Transactions of this Society, vol. 16 (1915), pp. 333–349; Cayley, *Collected Works*, vol. 11, p. 148.

In §3 we obtain, in terms of what we define as *higher Schwarzian derivatives*, an infinitude of differential and *integral* invariants under the inversive group. The fundamental differential inversive invariant, which is of the fifth order, was first obtained by G. Mullins[6] and later, using other methods, by Liebmann, Kubota, Morley, Patterson and Kasner.[7] The use of higher Schwarzian derivatives enables us to obtain simply and in compact form an infinitude of relative and absolute differential and integral invariants of curves.

In §4 we derive inversive invariants of horn angles of order $n \geqq 3$.

2. **Conformal invariants of horn angles of the first order.** Let the transformation

$$\mathrm{T}: \qquad z = \phi(u), \quad \bar{z} = \bar{\phi}(\bar{u}), \quad z = x + iy, \quad \bar{z} = x - iy$$

map a region r of the gaussian z-plane conformally on the region r' of the u-plane. Let the analytic arcs[8] c_1 and c_2 in the region r be the first and second sides respectively of a horn angle of the first order, α, the vertex of which is at the point P. Let T map c_1 and c_2 upon the arcs c_1' and c_2' respectively in r' where the latter form a horn angle of the first order, α', with vertex at P'. Let θ represent the common angle of inclination at P of c_1 and c_2, k_1 and k_2 their respective curvatures at P, while dk_1/ds_1 and dk_2/ds_2 represent respectively the values at P of the derivative of curvature with respect to arc-length. The same symbols primed represent the corresponding quantities associated with c_1' and c_2' at P'.

Let $f(z, \bar{z}) = 0$ be the self-conjugate equation (in isotropic coordinates) of a curve c in the z-plane. Likewise, let arcs c_1 and c_2 be represented by $f_1(z, \bar{z}) = 0$, $f_2(z, \bar{z}) = 0$, respectively, and let their images c', c_1', c_2' on the u-plane be given by $g(u, \bar{u}) = 0$, $g_1 = 0$, $g_2 = 0$, respectively. By $d^n z/d\bar{z}^n$, $d^n z_1/d\bar{z}_1^n$, $d^n z_2/d\bar{z}_2^n$ we designate the nth derivatives taken along c, c_1 and c_2 respectively. Likewise $d^n u/d\bar{u}^n$, $d^n u_1/d\bar{u}_1^n$ and $d^n u_2/d\bar{u}_2^n$ represent respectively derivatives along c', c_1' and c_2' in the u-plane.

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[6] Columbia University Dissertation, 1917.

[7] H. Liebmann, *Beiträge zur Inversionsgeometrie der Kurven*, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Munich, vol. 1 (1923), p. 79. T. Kubota, *Beiträge zur Inversionsgeometrie*, Tôhoku Imperial University Science Reports, vol. 13 (1924–1925), p. 243. F. Morley, *On differential inversive geometry*, American Journal of Mathematics, vol. 46 (1926), p. 144. B. Patterson, *The differential invariants of inversive geometry*, ibid., vol. 50 (1928), p. 553. E. Kasner, Transactions of this Society, vol. 44 (1938), p. 25.

[8] This restriction can be relaxed to require only that the arcs possess derivatives of the third order.

$$\frac{dz}{d\bar{z}} = \left(\frac{dz}{du} \Big/ \frac{d\bar{z}}{d\bar{u}}\right)\frac{du}{d\bar{u}} \tag{2.1}$$

we obtain

$$\frac{d^2z}{d\bar{z}^2} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)\left[\left(\frac{dz}{du} \Big/ \frac{d\bar{z}}{d\bar{u}}\right)\frac{d^2u}{d\bar{u}^2} + \left(\frac{du}{d\bar{u}}\right)^2 \frac{d}{du}\left(\frac{dz}{du} \Big/ \frac{d\bar{z}}{d\bar{u}}\right)\right]. \tag{2.2}$$

Evidently the derivatives of z with respect to u as well as of $\bar{z}$ with respect to $\bar{u}$ are independent of the particular curve c; furthermore, α' being a horn angle of the first order, $du_1/d\bar{u}_1 = du_2/d\bar{u}_2$ at P'; consequently, by applying equation (2.2) to c_i and c_i', and subtracting the results, we get

$$\frac{d^2z_2}{d\bar{z}_2^2} - \frac{d^2z_1}{d\bar{z}_1^2} = \left[\frac{dz}{du} \Big/ \left(\frac{d\bar{z}}{d\bar{u}}\right)^2\right]\left[\frac{d^2u_2}{d\bar{u}_2^2} - \frac{d^2u_1}{d\bar{u}_1^2}\right]. \tag{2.3}$$

Since $dz_1/d\bar{z}_1 = dz_2/d\bar{z}_2$ at P, division of equation (2.3) by equation (2.1) applied to c_i and c_i' results in

$$\frac{d^2z_2}{d\bar{z}_2^2} \Big/ \frac{dz_2}{d\bar{z}_2} - \frac{d^2z_1}{d\bar{z}_1^2} \Big/ \frac{dz_1}{d\bar{z}_1} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)\left[\frac{d^2u_2}{d\bar{u}_2^2} \Big/ \frac{du_2}{d\bar{u}_2} - \frac{d^2u_1}{d\bar{u}_1^2} \Big/ \frac{du_1}{d\bar{u}_1}\right]. \tag{2.4}$$

Consequently, *the left members of* (2.3) *and* (2.4) *represent complex relative conformal invariants of the second order of horn angles of the first order.*

When the variables in the Schwarzian derivative $\{y, x\}$ undergo a transformation $x = x(s)$, $y = y(t)$, it was proved by Cayley[9] that

$$\{y, x\} = \left(\frac{dt}{dx}\right)^2\{y, t\} - \left(\frac{ds}{dx}\right)^2\{x, s\} + \left(\frac{ds}{dx}\right)^2\{t, s\}. \tag{2.5}$$

In isotropic coordinates this becomes

$$\{z, \bar{z}\} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^2\left[\{u, \bar{u}\} - \{\bar{z}, \bar{u}\} + \{z, u\}\left(\frac{du}{d\bar{u}}\right)^2\right] \tag{2.5'}$$

showing the effect of a direct conformal transformation on $\{z, \bar{z}\}$.

[9] A. Cayley, "On the Schwarzian derivative and polyhedral functions," *Collected Works*, vol. 11, p. 148.

From (2.5′) we find that, for the horn angles α and α' at points P and P',

$$\{z_2, \bar{z}_2\} - \{z_1, \bar{z}_1\} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^2 [\{u_2, \bar{u}_2\} - \{u_1, \bar{u}_1\}]. \tag{2.6}$$

Consequently, *horn angles of the first order possess the complex relative conformal invariant of the third order given by the left member of* (2.6).

From the relative invariants (2.4) and (2.6) of index one and two respectively we obtain the *absolute invariant of the third order*

$$M_{12} = \frac{\left(\dfrac{d^2z_2}{d\bar{z}_2^2} \Big/ \dfrac{dz_2}{d\bar{z}_2} - \dfrac{d^2z_1}{d\bar{z}_1^2} \Big/ \dfrac{dz_1}{d\bar{z}_1}\right)^2}{\{z_2, \bar{z}_2\} - \{z_1, \bar{z}_1\}}.$$

A simple computation indicates that

$$\begin{gathered} \frac{dz}{d\bar{z}} = e^{2i\theta}, \qquad \frac{d^2z}{d\bar{z}^2} = 2ie^{3i\theta}k, \\ \frac{d^3z}{d\bar{z}^3} = 2ie^{4i\theta}\left(\frac{dk}{ds} + 3ik^2\right). \end{gathered} \tag{2.7}$$

Substitution of the values given by (2.7) in (2.4) and (2.6) shows that each of these complex relative invariants implies one real relative invariant, namely,

$$k_2 - k_1 \quad \text{and} \quad dk_2/ds_2 - dk_1/ds_1$$

respectively. In this form they were given in a joint paper by Kasner and Comenetz.[10]

Substitution from (2.7) in M_{12} leads us likewise to a single real invariant, namely, Kasner's unique absolute conformal invariant of the third order of horn angles of the first order:

$$M_{12}^* = \frac{(k_2 - k_1)^2}{dk_2/ds_2 - dk_1/s_1}, \qquad M_{12} = 2iM_{12}^*.$$

Comenetz[11] has proved that if a conformal transformation, S:

$$x_2 = \phi(x_1, y_1), \qquad y_2 = \psi(x_1, y_1)$$

maps a region r_1 in the x_1y_1-plane on a region r_2 in the x_2y_2-plane, so

[10] Kasner and Comenetz, National Academy of Sciences, loc. cit.

[11] American Mathematical Monthly, vol. 45 (1938), p. 86.

that a curve c_1 in r_1 is mapped on c_2 in r_2, then

$$k_2 = mk_1 + m_{y_1} \cos \theta_1 - m_{x_1} \sin \theta_1$$

and

$$dk_2/ds_2 = m^2 dk_1/ds_1 + m[m_{x_1y_1} \cos 2\theta_1 + \tfrac{1}{2}(m_{y_1y_1} - m_{x_1x_1}) \sin 2\theta_1],$$

where $m = 1/(J)^{1/2}$ and J is the jacobian of S. These formulas, describing the laws of transformation of curvature and rate of change of curvature under conformal mapping, are equivalent, except for factors, to (2.2) and (2.5′) respectively. Consequently, Comenetz's formulas are obtainable by appropriately interpreting standard identities that express the second derivative and Schwarzian derivative in terms of new variables. This remark enables us to regard (2.4), (2.6) and M_{12} as invariants of horn angles under a group other than the conformal group.

Explicitly, let the curves $f_1(x, y) = 0$, $f_2(x, y) = 0$ form a horn angle of the first order in a region R of the xy-plane, x, y being cartesian coordinates. Let the transformation $K: x = x(s)$, $y = y(t)$, map R uniformly and continuously on a region R' of the st-plane. We assume that K is provided with at least third derivatives in R. Then K maps the horn angle in R on a horn angle of the first order in R'. If we now write equations (2.1) and (2.2) in terms of x, y and s, t, thereby expressing the transformations that dy/dx and d^2y/dx^2 undergo under K, just as (2.5) does for $\{y, x\}$, we obtain the result that horn angles of the first order, under a group of transformations of type K, have the relative[12] invariants

$$A_{12} = \frac{d^2y_2}{dx_2^2} \Big/ \frac{dy_2}{dx_2} - \frac{d^2y_1}{dx_1^2} \Big/ \frac{dy_1}{dx_1}, \qquad B_{12} = \{y_2, x_2\} - \{y_1, x_1\}$$

and the absolute invariant A_{12}^2/B_{12}.

3. **Higher Schwarzian derivatives and inversive invariants of curves.** The inversive group in the gaussian plane is given by

$$z = (au + b)/(cu + d), \qquad z = \bar{u}, \qquad ad - bc \neq 0.$$

The first of these equations represents a subgroup G, containing only directly conformal inversive transformations. We first consider the transformations of G. Let any member of G transform the curve C:

[12] The term *relative* invariants as used here and elsewhere in this paper, means a quantity that is multiplied by some function which depends only upon the transformation; however, contrary to usual practice, the multiplier need not be a power of the jacobian of the transformation.

$f(z, \bar{z}) = 0$ into C': $g(u, \bar{u}) = 0$. Then, inasmuch as for such transformations $\{z, u\} = \{\bar{z}, \bar{u}\} = 0$, (2.5′) reduces to

$$\{z, \bar{z}\} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^2 \{u, \bar{u}\} \tag{3.1}$$

from which we gather that $\{z, \bar{z}\}$ *is a relative invariant of curves under* G. Noting that $d\bar{z} = (d\bar{z}/d\bar{u})d\bar{u}$, we obtain

$$\{z, \bar{z}\} d\bar{z}^2 = \{u, \bar{u}\} d\bar{u}^2$$

indicating that $\{z, \bar{z}\} d\bar{z}^2$ is an absolute invariant under G. Since

$$\{z, \bar{z}\} = 2ie^{2i\theta} \frac{dk}{ds}, \qquad d\bar{z} = e^{-i\theta} ds,$$

we get the real absolute invariant

$$(-1/2i)\{z, \bar{z}\} d\bar{z}^2 = -dkds.$$

If we let $d\lambda^2 = \pm dkds$, choosing the sign that makes $d\lambda^2$ positive, the integral invariant λ represents what has come to be known as the inversive arc-length.[13] Letting ρ represent radius of curvature,

$$-dkds = d\rho ds/\rho^2,$$

under which form this invariant was found, by other means, by Liebmann and Kubota.

Differentiating (3.1) twice, we obtain

$$\frac{d}{d\bar{z}} \{z, \bar{z}\} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^3 \frac{d}{d\bar{u}} \{u, \bar{u}\} - 2\{u, \bar{u}\} \frac{d^2\bar{z}}{d\bar{u}^2} \Big/ \left(\frac{d\bar{z}}{d\bar{u}}\right)^4 \tag{3.2}$$

and

$$\begin{aligned} \frac{d^2}{d\bar{z}^2} \{z, \bar{z}\} = {} & \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^4 \frac{d^2}{d\bar{u}^2} \{u, \bar{u}\} \\ & - 5 \frac{d}{d\bar{u}} \{u, \bar{u}\} \frac{d^2\bar{z}}{d\bar{u}^2} \Big/ \left(\frac{d\bar{z}}{d\bar{u}}\right)^5 \\ & - 2\{u, \bar{u}\} \left[\frac{d^3\bar{z}}{d\bar{u}^3} \Big/ \left(\frac{d\bar{z}}{d\bar{u}}\right)^5 \right. \\ & \left. - 4 \left(\frac{d^2\bar{z}}{d\bar{u}^2}\right)^2 \Big/ \left(\frac{d\bar{z}}{d\bar{u}}\right)^6 \right]. \end{aligned} \tag{3.3}$$

[13] See, e.g., Patterson, American Journal of Mathematics, loc. cit.

From (3.1), (3.2) and (3.3) we then obtain, recalling that $\{z, u\}=0$,

$$\frac{d^2}{d\bar{z}^2}\{z,\bar{z}\}/\{z,\bar{z}\}-\frac{5}{4}\left[\frac{d}{d\bar{z}}\{z,\bar{z}\}/\{z,\bar{z}\}\right]^2$$

$$=\left(1\Big/\frac{d\bar{z}}{d\bar{u}}\right)^2\left[\frac{d^2}{d\bar{u}^2}\{u,\bar{u}\}/\{u,\bar{u}\}\right. \tag{3.4}$$

$$\left.-\frac{5}{4}\left(\frac{d}{d\bar{u}}\{u,\bar{u}\}/\{u,\bar{u}\}\right)^2\right].$$

Let the left member of (3.4) be represented by the symbol $\{z, \bar{z}\}_2$, which we shall call *the second Schwarzian derivative of z with respect to* $\bar{z}$. It follows from (3.4) that $\{z, \bar{z}\}_2$ is a relative invariant of the fifth order under the direct inversive group. In our new notation (3.4) becomes

$$\{z,\bar{z}\}_2=\left(1\Big/\frac{d\bar{z}}{d\bar{u}}\right)^2\{u,\bar{u}\}_2. \tag{3.4'}$$

Generalizing the definition of the second Schwarzian derivative by a process of recurrence, we shall call

$$\{z,\bar{z}\}_{n+1}=\frac{d^2}{d\bar{z}^2}\{z,\bar{z}\}_n/\{z,\bar{z}\}_n-\frac{5}{4}\left(\frac{d}{d\bar{z}}\{z,\bar{z}\}_n/\{z,\bar{z}\}_n\right)^2 \tag{3.5}$$

the $(n+1)$-*th Schwarzian derivative*. We let $\{z, \bar{z}\}_1 \equiv \{z, \bar{z}\}$.

It can be shown by induction[14] that *the Schwarzian derivatives of order* $n \geqq 1$ *are all relative invariants of index two under the direct inversive group*, or that

$$\{z,\bar{z}\}_k=\left(1\Big/\frac{d\bar{z}}{d\bar{u}}\right)^2\{u,\bar{u}\}_k, \qquad k \geqq 1. \tag{3.6}$$

Consequently, *the ratios*

$$\{z,\bar{z}\}_j/\{z,\bar{z}\}_k, \qquad j \neq k;\ j, k = 1, 2, 3, \cdots, \tag{3.7}$$

are absolute differential invariants of plane curves under G. Of these, the one of lowest order, namely that of the fifth order, is

$$I_5=\{z,\bar{z}\}_2/\{z,\bar{z}\}.$$

We proceed to express I_5 in terms of k and derivatives of k with respect to s.

[14] The proof of this statement involves straightforward computation, which we shall omit.

From

$$\{z, \bar{z}\} = 2ie^{2i\theta}dk/ds$$

we obtain

$$\frac{d}{d\bar{z}}\{z, \bar{z}\} = 2e^{3i\theta}\left(i\frac{d^2k}{ds^2} - 2k\frac{dk}{ds}\right)$$

and

$$\frac{d^2}{d\bar{z}^2}\{z, \bar{z}\} = 2e^{4i\theta}\left[i\frac{d^3k}{ds^3} - 5k\frac{d^2k}{ds^2} - 6ik^2\frac{dk}{ds} - 2\left(\frac{dk}{ds}\right)^2\right].$$

Substituting these expressions in $\{z, \bar{z}\}_2$, we find

$$\{z, \bar{z}\}_2 = e^{2i\theta}\left[\frac{d^3k}{ds^3}\Big/\frac{dk}{ds} - \frac{5}{4}\left(\frac{d^2k}{ds^2}\Big/\frac{dk}{ds}\right)^2 - k^2 + 2i\frac{dk}{ds}\right] \tag{3.8}$$

and therefore

$$I_5 = 1 - \frac{i}{2}\left[\frac{d^3k}{ds^3}\Big/\left(\frac{dk}{ds}\right)^2 - \frac{5}{4}\left(\frac{d^2k}{ds^2}\right)^2\Big/\left(\frac{dk}{ds}\right)^3 - k^2\Big/\frac{dk}{ds}\right].$$

Hence, I_5 implies only one real invariant, namely

$$I_5^* = \frac{d^3k}{ds^3}\Big/\left(\frac{dk}{ds}\right)^2 - \frac{5}{4}\left(\frac{d^2k}{ds^2}\right)^2\Big/\left(\frac{dk}{ds}\right)^3 - k^2\Big/\frac{dk}{ds}.$$

In terms of ρ and derivatives of ρ with respect to s,

$$I_5^* = \left[-4\frac{d^3\rho}{ds^3}\frac{d\rho}{ds}\rho^2 + 4\frac{d^2\rho}{ds^2}\left(\frac{d\rho}{ds}\right)^2\rho - 4\left(\frac{d\rho}{ds}\right)^4 + 5\left(\rho\frac{d^2\rho}{ds^2}\right)^2 + 4\left(\frac{d\rho}{ds}\right)^2\right]\Big/\left[4\left(\frac{d\rho}{ds}\right)^3\right]$$

in which form this invariant was first obtained by G. Mullins.[15]

Recalling that the inversive arc length $\lambda = \int(\pm dk/ds)^{1/2}ds$ and that therefore $dk/ds = (d\lambda/ds)^2$ we find that I_5^* assumes the simple form

$$I_5^* = (2\{\lambda, s\} - k^2)/\left(\frac{d\lambda}{ds}\right)^2.$$

From (3.6) it obviously follows that

$$\{z, \bar{z}\}_k d\bar{z}^2 = \{u, \bar{u}\}_k d\bar{u}^2, \qquad k \geq 1.$$

[15] Columbia University Dissertation, 1917.

Consequently,

$$\lambda_k = \int (\{z, \bar{z}\}_k)^{1/2} d\bar{z}$$

is an integral invariant under G. However, λ_k, being complex, implies two real invariants, one of which may be only a constant. When $k=1$ the unique non-constant invariant, as shown above, is the inversive arc-length.

Under the complete inversive group, which includes the group G and the involutory transformation $z=\bar{u}$, the absolute values of the invariants referred to above are invariant.

4. **Inversive invariants of horn angles.** From (3.6) it obviously follows that curvilinear angles, and therefore horn angles of order n, $n \geqq 1$, have, under the direct inversive group G, the absolute invariants of odd order, namely $2k+1$:

$$\{z_2, \bar{z}_2\}_k / \{z_1, \bar{z}_1\}_k, \qquad k \geqq 1.$$

However, we can obtain from (3.6) certain other, less obvious horn angle invariants. Let α be a horn angle of order $2n+1$, $n \geqq 1$. Differentiation of (3.6) results in

$$\frac{d}{d\bar{z}} \{z, \bar{z}\}_n = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^3 \frac{d}{d\bar{u}} \{u, \bar{u}\}_n - 2\{u, \bar{u}\}_n \frac{d^2\bar{z}}{d\bar{u}^2} \Big/ \left(\frac{d\bar{z}}{d\bar{u}}\right)^4. \tag{4.1}$$

Since α is of order $2n+1$, $\{z_1, \bar{z}_1\}_n = \{z_2, \bar{z}_2\}_n$, $\{u_1, \bar{u}_1\}_n = \{u_2, \bar{u}_2\}_n$ at the vertex of the horn angle; consequently, by subtracting the equations obtained by stating (4.1) for the two sides of α we find that

$$\frac{d}{d\bar{z}_2} \{z_2, \bar{z}_2\}_n - \frac{d}{d\bar{z}_1} \{z_1, \bar{z}_1\}_n = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^3 \left[\frac{d}{d\bar{u}_2} \{u_2, \bar{u}_2\}_n - \frac{d}{d\bar{u}_1} \{u_1, \bar{u}_1\}_n\right]. \tag{4.2}$$

Therefore, *the left member of* (4.2) *represents a relative inversive invariant of order* $2n+2$ *and index three of horn angles of order* $2n+1$, $n \geqq 1$.

Let α be a horn angle of even order: $2n+2$, $n \geqq 1$. Now it follows from (3.6), by subtraction of (3.5) stated for the two sides of α that

$$\frac{d^2}{d\bar{z}_2^2}\{z_2, \bar{z}_2\}_n - \frac{d^2}{d\bar{z}_1^2}\{z_1, \bar{z}_1\}_n = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^4 \left[\frac{d^2}{d\bar{u}_2^2}\{u_2, \bar{u}_2\}_n - \frac{d^2}{d\bar{u}_1^2}\{u_1, \bar{u}_1\}_n\right]. \tag{4.3}$$

Consequently, *horn angles of order* $2n+2$, $n \geqq 1$, *have, under the direct inversive group, the relative invariant of order* $2n+3$ *and index four given by* (4.3).

BROOKLYN COLLEGE

ON A PROPERTY OF k CONSECUTIVE INTEGERS[1]

ALFRED BRAUER

S. S. Pillai[2] has just proved the following theorem: In every set of less than 17 consecutive integers there exists at least one integer which is relatively prime to all the others; there are sequences of k integers for $k=17, 18, \cdots, 430$, however, which have not this property. Pillai conjectures that the same is valid for every $k \geqq 17$. I shall prove that this conjecture is true.

The method of the proof is similar to the method I applied in a joint paper with H. Zeitz[3] in proving that the following conjecture is wrong for every prime $p \geqq 43$.

Denote by p_n the nth prime. Then there exist at most $2p_{n-1}-1$ consecutive integers such that each of these integers is at least divisible by one of the primes $p_1, p_2, \cdots, p_n$.

This conjecture was used by Legendre for his proof of the theorem of the primes in arithmetical progressions. First I prove the following.

LEMMA. *Let $\pi(x)$ be the number of primes $p \leqq x$. Then we have*

$$\pi(2x) - \pi(x) \geqq 2\left[\frac{\log x}{\log 2}\right] + 2 \tag{1}$$

for every $x \geqq 75$.

PROOF. If we put, as usual,

$$\vartheta(x) = \sum_{p \leqq x} \log p,$$

then we have

$$\begin{aligned} \pi(2x) - \pi(x) &= \sum_{x<p\leqq 2x} 1 \geqq \sum_{x<p\leqq 2x} (\log p/\log 2x) \\ &= \left\{ \sum_{x<p\leqq 2x} \log p \right\} /\log 2x = \{\vartheta(2x) - \vartheta(x)\}/\log 2x. \end{aligned} \tag{2}$$

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[1] Presented to the Society, September 12, 1940.

[2] S. S. Pillai, *On m consecutive integers*, Proceedings of the Indian Academy of Sciences, section A, vol. 11 (1940), pp. 6–12.

[3] A. Brauer und H. Zeitz, *Über eine zahlentheoretische Behauptung von Legendre*, Sitzungsberichte der Berliner mathematischen Gesellschaft, vol. 29 (1930), pp. 116–125. Cf. A. Brauer, *Question concerning the maximum term in the diatomic series—proposed by A. A. Bennett*, American Mathematical Monthly, vol. 40 (1933), pp. 409–410.

It is known that[4]

$$\vartheta(2x) - \vartheta(x) \geqq \vartheta(2x-2) - \vartheta(x) > .7x - 3.4x^{1/2} - 4.5\log^2 x - 24\log x - 32. \tag{3}$$

Hence, by (1), (2), and (3), it is sufficient to prove that

$$.7x - 3.4x^{1/2} - 4.5\log^2 x - 24\log x - 32 > (2\log x/\log 2 + 2)(\log x + \log 2),$$

$$f(x) = .7x - 3.4x^{1/2} - \log^2 x(4.5 + 2/\log 2) - 28\log x - 32 - 2\log 2 > 0.$$

It is easy to see that $f(x) > 0$ holds for $x = 1024$, since $\log 1024 < 7$. Moreover we have

$$f'(x) = .7 - 1.7x^{-1/2} - \frac{9 + 4/\log 2}{x}\log x - \frac{28}{x} > 0 \quad \text{for } x \geqq 1024.$$

Hence $f(x)$ is increasing for $x \geqq 1024$ and the lemma is proved for $x \geqq 1024$.

For $75 \leqq x < 1024$ the lemma can be proved directly. For instance, it follows for $591 \leqq x < 1024$ and for $355 \leqq x < 591$ by the fact that there are 22 primes between 1024 and 1182 and 20 primes between 591 and 710. In the same way we get the lemma for $231 \leqq x < 355$, $159 \leqq x < 231$, and so on.

THEOREM. *For every $k \geqq 17$ there exists a sequence of k consecutive integers such that none of these k integers is relatively prime to the product of the others.*

PROOF. In view of the paper of Pillai, it is sufficient to prove the theorem for $k \geqq 300$. We put

$$m = \left[\frac{k}{4}\right] \geqq 75. \tag{4}$$

Let $p_1, p_2, \cdots, p_r$ be the primes in the closed interval $\{1 \cdots m\}$ and $p_{r+1}, p_{r+2}, \cdots, p_s$ the primes in the closed interval $\{m+1 \cdots 2m\}$. If we consider k consecutive integers, then each of the primes

$$p_1, p_2, \cdots, p_r, p_{r+1}, p_{r+2}, \cdots, p_s \tag{5}$$

divides at least two of the k integers, since each of these primes is less than $2m$, hence by (4) less than $k/2$. Therefore each of these k integers which is divisible by at least one of the primes (5) is not relatively prime to all the $k-1$ other integers. Hence it is sufficient to

[4] See, for example, E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1, 1909, p. 91.

prove that there exist sequences of k integers such that for $k \geqq 300$ each of these integers is divisible by at least one of the primes (5).

We consider the simultaneous congruences

$$x \equiv 1 \pmod 2, \qquad x \equiv 0 \pmod{p_2 p_3 \cdots p_r}. \tag{6}$$

Let x be a solution of (6). Then the integers

$$x-2m,\ x-2m+2,\ \cdots,\ x-2,\ x,\ x+2,\ \cdots,\ x+2m-2,\ x+2m \tag{7}$$

form a sequence of $2m+1$ odd integers of the form

$$x \pm 2\mu, \qquad \mu = 0, 1, \cdots, m. \tag{8}$$

If μ is divisible by the odd prime p_ν, we have $p_\nu \leqq p_r$, since $\mu \leqq m$ because of (8). Hence we obtain from (6) that

$$x \pm 2\mu \equiv 0 \pmod{p_\nu}. \tag{9}$$

It follows from (9) that all those integers of (7) which have not the form $x \pm 2^\tau$ with $\tau \geqq 1$ are divisible by at least one of the primes $p_2, p_3, \cdots, p_r$.

If we put

$$\left[\frac{\log m}{\log 2}\right] + 1 = t, \tag{10}$$

then the integers of the form $x \pm 2^\tau$ with $\tau \geqq 1$ in the set (7) are the integers

$$x \pm 2,\ x \pm 2^2,\ \cdots,\ x \pm 2^t. \tag{11}$$

By (4), it follows from the lemma and from (10) that the number of primes in the closed interval $\{m+1 \cdots 2m\}$ is

$$\pi(2m) - \pi(m) \geqq 2\left[\frac{\log m}{\log 2}\right] + 2 = 2t.$$

On the other hand the primes in this interval were $p_{r+1}, p_{r+2}, \cdots, p_s$, hence

$$s - r \geqq 2t, \qquad p_{r+2t} \leqq p_s. \tag{12}$$

Beside the congruences (6) we now subject x to the following $2t$ congruences

$$\begin{aligned} x + 2^\tau &\equiv 0 \pmod{p_{r+\tau}}, \\ x - 2^\tau &\equiv 0 \pmod{p_{r+t+\tau}}, \end{aligned} \qquad \tau = 1, 2, \cdots, t. \tag{13}$$

These congruences always have solutions. For every solution x all the

numbers (7) are divisible by at least one of the primes (5), since each of the integers (11) is divisible by at least one of the primes $p_{r+1}, p_{r+2}, \cdots, p_s$ because of (13) and (12).

Hence each of the $4m+3$ consecutive integers

$$(14)\quad x-2m-1,\ x-2m,\ x-2m+1,\ \cdots,\ x-1,\ x,\ x+1,\ \cdots,\ x+2m+1$$

is divisible by at least one of the primes (5), since

$$x - 2m - 1 \equiv x - 2m + 1 \equiv \cdots \equiv x - 1 \equiv x + 1 \equiv \cdots \equiv x + 2m + 1 \equiv 0 \pmod 2.$$

Because of (4) we have

$$k \leqq 4m + 3.$$

Therefore we can take k consecutive integers from (14). None of these k integers is relatively prime to the product of the $k-1$ others.

INSTITUTE FOR ADVANCED STUDY

A CORRECTION TO "A REMARK ON THE SUM AND THE INTERSECTION OF TWO NORMAL IDEALS IN AN ALGEBRA"

TADASI NAKAYAMA

K. Asano has called my attention to an error in my paper *A remark on the sum and the intersection of two normal ideals in an algebra*.[1] Theorem 1 of the paper holds only for local simple algebras, while the remaining theorems are true as they stand.

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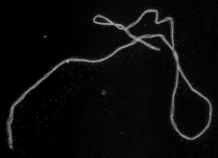
[1] This Bulletin, vol. 46 (1940), p. 469